1 Subgradients

Theorem (Moreau-Rockafellar): Let $f, g : \mathbb{R}^n \to (-\infty, \infty]$ be proper convex functions. Then for every $x_0 \in \mathbb{R}^n$

$$\partial f(x_0) + \partial g(x_0) \subset \partial(f+g)(x_0)$$

Moreover, suppose int $\text{dom}(f) \cap \text{dom}(g) \neq \emptyset$. Then for every $x_0 \in \mathbb{R}^n$,

$$\partial f(x_0) + \partial g(x_0) = \partial(f+g)(x_0)$$

Proof. Let $u_1 \in \partial f(x_0),\ u_2 \in \partial g(x_0)$. Then for every $x \in \mathbb{R}^n$,

$$f(x) \geq f(x_0) + \langle u_1, x - x_0 \rangle,\ g(x) \geq g(x_0) + \langle U_2, x - x_0 \rangle$$

Hence, adding the two inequalities shows that $u + v \in \partial(f+g)(x_0)$.
Now, let $v \in \partial(f+g)(x_0)$. Note that $f(x_0) \neq \infty$, otherwise this implies that $f + g \equiv \infty$. Similarly, $g(x_0) \neq \infty$. Next, consider the following two sets

$$\Lambda_f := \{(x - x_0, y) :\ y > f(x) - f(x_0) - \langle v, x - x_0 \rangle\}$$
$$\Lambda_g := \{(x - x_0, y) :\ -y \geq g(x) - g(x_0)\}.$$

Λ_f, Λ_g are both nonempty and convex (consider epi(f), epi(g)). Also, since $v \in \partial(f+g)(x_0),\ \Lambda_f \cap \Lambda_g = \emptyset$ (otherwise, adding the above two inequalities contradict the fact that $v \in \partial(f+g)$)
Then Λ_f, Λ_g can be separated by a hyperplane. So there exists $(a, b) \neq 0, c$ such that

$$\langle a, x - x_0 \rangle + by \leq c,\ \forall (x, y) \text{ such that } y > f(x) - f(x_0) - \langle v, x - x_0 \rangle$$

$$\langle a, x - x_0 \rangle + by \geq c,\ \forall (x, y) \text{ such that } -y \geq g(x) - g(x_0)$$

Since $(0, 0) \in \Lambda_g$, $c \leq 0$. Since $(0, 1) \in \Lambda_f, b \leq 0$.
For all $\epsilon > 0, (0, \epsilon) \in \Lambda_f$, since $b \leq 0$, letting $\epsilon \to 0$, we get $c \geq 0$. Hence $c = 0$.
Suppose $b = 0$, we have

$$\langle a, x - x_0 \rangle \leq 0,\ \forall (x, y) \text{ such that } y > f(x) - f(x_0) - \langle v, x - x_0 \rangle$$

$$\langle a, x - x_0 \rangle \geq 0,\ \forall (x, y) \text{ such that } -y \geq g(x) - g(x_0)$$

which are equivalent to

$$\langle a, x - x_0 \rangle \leq 0,\ \forall x \in \text{dom}(f)$$

$$\langle a, x - x_0 \rangle \geq 0, \ \forall x \in \text{dom}(g)$$

Let $\overline{x} \in \text{int dom}(f) \cap \text{dom}(g)$. Then $\langle a, \overline{x} - x_0 \rangle = 0$. Since $\overline{x} \in \text{int dom}(f)$, there exists $\delta > 0$ such that $B(\overline{x}, \delta) \subset \text{dom}(f)$. Then

$$\langle a, \frac{\delta a}{2} \rangle = \langle a, \overline{x} + \frac{\delta a}{2} - x_0 \rangle \leq 0$$

So $a = 0$. This contradicts the fact that $(a, b) \neq 0$. Hence $b < 0$.
Let $-u_2 = \frac{a}{-b}$, we have

$$\langle -u_2, x - x_0 \rangle \leq y, \ \forall (x, y) \text{ such that } y > f(x) - f(x_0) - \langle v, x - x_0 \rangle.$$

$$\langle -u_2, x - x_0 \rangle \geq y, \forall (x, y) \text{ such that } -y \geq g(x) - g(x_0)$$

Consider $y = g(x_0) - g(x)$, then $u_2 \in \partial g(x_0)$.
By considering $(x, f(x) - f(x_0) - \langle v, x - x_0 \rangle + \epsilon$ and letting $\epsilon \to 0$, we have $u_1 = v - u_2 \in \partial f(x_0)$.
Hence $v = u_1 + u_2 \in \partial f(x_0) + \partial g(x_0)$.
Therefore $\partial (f + g)(x_0) \subset \partial f(x_0) + \partial g(x_0)$. □