1 Relative Interior

Definition: (Relative interior) Let $C \subset \mathbb{R}^n$. We say that $x$ is a relative interior point of $C$ if $x \in B(x, \epsilon) \cap \text{aff}(C) \subset C$, for some $\epsilon > 0$. The set of all relative interior point of $C$ is called the relative interior of $C$, and is denoted by $\text{ri}(C)$. The relative boundary of $C$ is equal to $\text{cl}(C) \setminus \text{ri}(C)$.

Proposition 1.3.2: (Line Segment Property) Let $C$ be a nonempty convex set. If $x \in \text{ri}(C)$, $\overline{x} \in \text{cl}(C)$, then $\alpha x + (1 - \alpha)\overline{x} \in \text{ri}(C)$ for $\alpha \in (0, 1]$.

Proof. Fix $\alpha \in (0, 1]$. Consider $x_\alpha = \alpha x + (1 - \alpha)\overline{x}$. Since $\overline{x} \in \text{cl}(C)$, for all $\epsilon > 0$, we have $\overline{x} \in C + (B(0, \epsilon) \cap \text{aff}(C))$. Then
\[
B(x_\alpha, \epsilon) \cap \text{aff}(C) = \{\alpha x + (1 - \alpha)\overline{x} + (B(0, \epsilon) \cap \text{aff}(C))
\subset \{\alpha x\} + (1 - \alpha)C + (2 - \alpha)(B(0, \epsilon) \cap \text{aff}(C))
= (1 - \alpha)C + \alpha \left[B \left(x, \frac{2 - \alpha}{\alpha} \epsilon \right) \cap \text{aff}(C)\right]
\]
Since $x \in \text{ri}(C)$, $B \left(x, \frac{2 - \alpha}{\alpha} \epsilon \right) \cap \text{aff}(C) \subset C$, for sufficiently small $\epsilon$.
So $B(x_\alpha, \epsilon) \cap \text{aff}(C) \subset \alpha C + (1 - \alpha)C = C$ (since $C$ is convex). Therefore, $x_\alpha \in \text{ri}(C)$.

Proposition 1.3.3: (Prolongation Lemma) Let $C$ be a nonempty convex set. Then we have
\[
x \in \text{ri}(C) \iff \forall \overline{x} \in C, \exists \gamma > 0 \text{ such that } x + \gamma(x - \overline{x}) \in C.
\]
In other words, $x$ is a relative interior point iff every line segment in $C$ having $x$ as one of the endpoints can be prolonged beyond $x$ without leaving $C$.

Proof. Suppose the condition holds for $x$. Let $\overline{x} \in \text{ri}(C)$. If $x = \overline{x}$, then we are done. So assume $x \neq \overline{x}$. Then there exists $\gamma > 0$ such that $y = x + \gamma(x - \overline{x}) \in C$. Hence $x = \frac{1}{1 + \gamma}y + \frac{\gamma}{1 + \gamma}\overline{x}$. Since $\overline{x} \in \text{ri}(C)$, $y \in C$, by the line segment property, we have $x \in \text{ri}(C)$. The other direction is clear from the fact that $x \in \text{ri}(C)$.

\[\square\]