1 Relative Interior

Definition: (Relative interior) Let $C \subset \mathbb{R}^n$. We say that x is a *relative interior point* of C if $x \in B(x,\epsilon) \cap \text{aff}(C) \subset C$, for some $\epsilon > 0$. The set of all relative interior point of C is called the *relative interior* of C, and is denoted by $\text{ri}(C)$. The *relative boundary* of C is equal to $\text{cl}(C) \setminus \text{ri}(C)$.

Proposition 1.3.2: (Line Segment Property) Let C be a nonempty convex set. If $x \in \text{ri}(C)$, $\overline{x} \in \text{cl}(C)$, then $\alpha x + (1-\alpha)\overline{x} \in \text{ri}(C)$ for $\alpha \in (0,1]$.

Proof. Fix $\alpha \in (0.1]$. Consider $x_\alpha = \alpha x + (1-\alpha)\overline{x}$. Since $\overline{x} \in \text{cl}(C)$, for all $\epsilon > 0$, we have $\overline{x} \in C + (B(0,\epsilon) \cap \text{aff}(C))$. Then

$$\begin{aligned} B(x_\alpha, \epsilon) \cap \text{aff}(C) &= \{\alpha x + (1-\alpha)\overline{x}\} + (B(0,\epsilon) \cap \text{aff}(C)) \\ &\subset \{\alpha x\} + (1-\alpha)C + (2-\alpha)(B(0,\epsilon) \cap \text{aff}(C)) \\ &= (1-\alpha)C + \alpha\left[B\left(x, \frac{2-\alpha}{\alpha}\epsilon\right) \cap \text{aff}(C)\right] \end{aligned}$$

Since $x \in \text{ri}(C)$, $B\left(x, \frac{2-\alpha}{\alpha}\epsilon\right) \cap \text{aff}(C) \subset C$, for sufficiently small ϵ.
So $B(x_\alpha, \epsilon) \cap \text{aff}(C) \subset \alpha C + (1-\alpha)C = C$ (since C is convex). Therefore, $x_\alpha \in \text{ri}(C)$.

□

Proposition 1.3.3: (Prolongation Lemma) Let C be a nonempty convex set. Then we have

$$x \in \text{ri}(C) \iff \forall \overline{x} \in C,\ \exists \gamma > 0 \text{ such that } x + \gamma(x - \overline{x}) \in C.$$

In other words, x is a relative interior point iff every line segment in C having x as one of the endpoints can be prolonged beyond x without leaving C.

Proof. Suppose the condition holds for x. Let $\overline{x} \in \text{ri}(C)$. If $x = \overline{x}$, then we are done. So assume $x \neq \overline{x}$. Then there exists $\gamma > 0$ such that $y = x + \gamma(x - \overline{x}) \in C$. Hence $x = \frac{1}{1+\gamma}y + \frac{\gamma}{1+\gamma}\overline{x}$. Since $\overline{x} \in \text{ri}(C)$, $y \in C$, by the line segment property, we have $x \in \text{ri}(C)$. The other direction is clear from the fact that $x \in \text{ri}(C)$. □