1 Basic concepts of convex optimization

In convex optimization, we consider the problem

$$\min_{x \in C} f(x)$$

where $f : \mathbb{R}^n \to (-\infty, \infty]$ is a convex function and $C$ is a convex subset of $\mathbb{R}^n$.

If $x \in C \cap \text{dom}(f)$, then $x$ is called feasible. If there is at least one feasible point, then the problem is called feasible.

$x^*$ is called a minimum of $f$ over $C$ if

$$x^* \in C \cap \text{dom}(f), \quad f(x^*) = \inf_{x \in C} f(x)$$

We may write $x^* \in \text{arg min}_{x \in C} f(x)$ or even $x^* = \text{arg min}_{x \in C} f(x)$ if $x^*$ is the unique minimizer.

Other than global minimum, we also have a weaker definition of local minimum, one that is only minimum compared to the point nearby. We call $x^*$ a local minimum of $f$ over $C$ if $x^* \in C \cap \text{dom}(f)$ and there exists $\epsilon > 0$ such that

$$f(x^*) \leq f(x), \quad \forall x \in C \text{ with } ||x - x^*|| < \epsilon$$

In the convex setting, we have the following nice result.

**Proposition:** Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a convex function and let $C$ be a convex set.

Then a local minimum of $f$ over $C$ is also a global minimum of $f$ over $C$.

If $f$ is strictly convex, then there exists at most one global minimum of $f$ over $C$.

**Existence of solution**

Consider the problem

$$\min_{x \in \mathbb{R}^n} f(x)$$

where $f$ is convex.

Suppose the level sets $V_a = \{x \mid f(x) \leq a\}$ are also compact. Then we can consider the problem

$$\min_{x \in V_a} f(x)$$
for some $V_0$ that is nonempty. Then there exist at least one global minimizer.

Remark: We can also show that $f$ is coercive, which is equivalent to the level sets of $f$ are compact.

1.1 Optimal conditions

In an unconstrained problem, one has a simple optimality test, which is the 'derivative' test in calculus.

Let $f$ be a differentiable convex function on $\mathbb{R}^n$. Then $x^*$ solves

\[
\min_{x \in \mathbb{R}^n} f(x)
\]

if and only if $\nabla f(x^*) = 0$.

How about a constrained problem? Let’s consider the general constrained problem

\[
\min_{x \in C} f(x)
\]

where $C$ is a convex set, and $f$ is convex.

We have the following result.

**Proposition:** Let $C$ be a nonempty convex set and let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex differentiable function over an open set that contains $C$. Then $x^* \in C$ minimizes $f$ over $C$ if and only if

\[
\langle \nabla f(x^*), (z - x^*) \rangle \geq 0, \quad \forall z \in C.
\]

**Proof.** Suppose $\langle \nabla f(x^*), (z - x^*) \rangle \geq 0, \forall z \in C$, then we have,

\[
f(z) - f(x^*) \geq \langle \nabla f(x^*), (z - x^*) \rangle \geq 0, \forall z \in C.
\]

Hence $x^*$ indeed minimizes $f$ over $C$.

Conversely, suppose $x^*$ minimizes $f$ over $C$. Suppose on the contrary that $\langle \nabla f(x^*), (z - x^*) \rangle < 0$ for some $z \in C$, then

\[
\lim_{\alpha \downarrow 0} \frac{f(x^* + \alpha(z - x^*)) - f(x^*)}{\alpha} = \langle \nabla f(x^*), (z - x^*) \rangle < 0.
\]

Then for sufficiently small $\alpha$, we have $f(x^* + \alpha(z - x^*)) - f(x^*) < 0$, contradicting the optimality of $x^*$. \qed
1.2 Examples

(a) Let’s consider the following linear constrained problem.

\[
\min_{x \in \mathbb{R}^n} f(x) \text{ subject to } Ax = b
\]

where \( A \) is a \( m \times n \) matrix and \( b \in \mathbb{R}^m \).

Suppose we have a solution \( x^* \), then

\[
\langle \nabla f(x^*), y - x^* \rangle \geq 0, \forall y \text{ such that } Ay = b
\]

This is the same as

\[
\langle \nabla f(x^*), h \rangle \geq 0, \forall h \in \text{Null}(A).
\]

Since \( -h \in \text{Null}(A) \) if \( h \in \text{Null}(A) \), we have

\[
\langle \nabla f(x^*), h \rangle = 0, \forall h \in \text{Null}(A).
\]

Hence \( \nabla f(x^*) \in \text{Null}(A)^\perp = \text{Ran}(A^T) \).

So there exists \( \mu \in \mathbb{R}^m \) such

\[
\nabla f(x^*) + A^T \mu = 0.
\]

To conclude, \( x^* \) is a solution to the minimization problem if and only if

1. \( Ax^* = b \)
2. There exists \( \mu^* \in \mathbb{R}^m \) such that \( \nabla f(x^*) + A^T \mu = 0 \).

(b) Let’s consider the minimization problem

\[
\min_{x \in \mathbb{R}^n} f(x), \text{ subject to } x \geq 0.
\]

Suppose we have a solution \( x^* \), then

\[
\langle \nabla f(x^*), y - x^* \rangle \geq 0, \forall y \in \mathbb{R}^n_+.
\]

In particular, \( 0, 2x^* \in \mathbb{R}^n_+ \), so

\[
\langle \nabla f(x^*), x^* \rangle = 0, \langle \nabla f(x^*), y \rangle \geq 0, \forall y \in \mathbb{R}^n_+.
\]

Hence, \( \nabla f(x^*) \geq 0 \). This is the same as saying there exists \( \lambda^* \geq 0 \) such that

\[
\nabla f(x^*) - \lambda^* = 0
\]

To conclude, \( x^* \) is a solution if and only if

1. \( x^* \geq 0 \)
2. There exists \( \lambda^* \geq 0 \) such that \( \nabla f(x^*) - \lambda^* = 0 \)
3. \( \lambda^*_i x^*_i = 0 \)