1 Normal cone

Let $C \subset \mathbb{R}^n$ be a convex set with $\overline{x} \in C$. The *normal cone* to C at $\overline{x}$ is

$$N(\overline{x}; C) := \{v \in \mathbb{R}^n |\ \langle v, x - \overline{x}\rangle \leq 0,\ \forall x \in C\}$$

Proposition: Let $\overline{x} \in C$, where C is a convex subset of ^{n}. Then we have:

1. $N(\overline{x}; C)$ is a closed convex cone containing the origin.
2. If $\overline{x}$ is a interior point, then $N(\overline{x}; C) = \{0\}$.

Consider a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^p$. The *adjoint mapping* $A^* : \mathbb{R}^p \to \mathbb{R}^n$ is defined by

$$\langle Ax, y\rangle = \langle x, A^* y\rangle,\ \forall x \in \mathbb{R}^n,\ y \in \mathbb{R}^p$$

Proposition: Let $B : \mathbb{R}^n \to \mathbb{R}^p$ be defined by $B(x) = Ax + b$, where A is linear. Given $c \in \mathbb{R}$, consider

$$C := \{x \in \mathbb{R}^n |\ Bx = c\}$$

For and $\overline{x} \in C$, we have

$$N(\overline{x}; C) = \{v \in \mathbb{R}^n |\ v = A^* y,\ y \in \mathbb{R}^p\} = \text{im}(A^*)$$

Proof. Let $v \in N(\overline{x}; C)$. Let $u \in \ker(A)$, then $A(\overline{x} - u) + b = A\overline{x} + b = c$. Hence $\langle v, u\rangle \geq 0$ for all $u \in \ker(A)$. But $-u \in \ker(A)$, so $\langle v, u\rangle = 0$ for all $u \in \ker(A)$.
Suppose there is no $y \in \mathbb{R}^p$ with $v = A^* y$. So $v \notin A^*(\mathbb{R}^p) := W$. Since W is nonempty, closed and convex, there exists a nonzero $\overline{u}$ such that

$$\sup\{\langle \overline{u}, w\rangle |\ w \in W\} < \langle \overline{u}, v\rangle.$$

Since $0 \in W$, so $\langle \overline{u}, v\rangle > 0$.
Also $\langle \overline{u}, A^*(ty)\rangle < \langle \overline{u}, v\rangle,\ \forall t \in \mathbb{R},\ \forall y \in \mathbb{R}^p$. Therefore $\langle \overline{u}, A^* y\rangle = 0$ (otherwise, we can choose t so that the above inequality doesn't hold).
Therefore, $\langle A\overline{u}, y\rangle = 0$ for all y. So $A\overline{u} = 0$. Hence $\overline{u} \in \ker A$ and $\langle v, \overline{u}\rangle > 0$. This is a contradiction.
Therefore, $N(\overline{x}; C) \subset \text{im}(A^*)$.
Conversely, suppose $v = A^* y$. For any $x \in C$, we have

$$\langle v, x - \overline{x}\rangle = \langle A^* y, x - \overline{x}\rangle = \langle y, Ax - A\overline{x}\rangle = 0$$

Hence $v \in N(\overline{x}; C)$. □