1 Normal cone

Let $C \subset \mathbb{R}^n$ be a convex set with $x \in C$. The normal cone to $C$ at $x$ is
\[
N(x; C) := \{ v \in \mathbb{R}^n | \langle v, x - x \rangle \leq 0, \ \forall x \in C \}
\]

**Proposition:** Let $x \in C$, where $C$ is a convex subset of $\mathbb{R}^n$. Then we have:

1. $N(x; C)$ is a closed convex cone containing the origin.
2. If $x$ is an interior point, then $N(x; C) = \{0\}$.

Consider a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^p$. The adjoint mapping $A^* : \mathbb{R}^p \to \mathbb{R}^n$ is defined by
\[
\langle Ax, y \rangle = \langle x, A^*y \rangle, \ \forall x \in \mathbb{R}^n, \ y \in \mathbb{R}^p
\]

**Proposition:** Let $B : \mathbb{R}^n \to \mathbb{R}^p$ be defined by $B(x) = Ax + b$, where $A$ is linear. Given $c \in \mathbb{R}$, consider
\[
C := \{ x \in \mathbb{R}^n | Bx = c \}
\]
For and $x \in C$, we have
\[
N(x; C) = \{ v \in \mathbb{R}^n | v = A^*y, \ y \in \mathbb{R}^p \} = \text{im}(A^*)
\]

**Proof.** Let $v \in N(x; C)$. Let $u \in \ker(A)$, then $A(x - u) + b = Ax + b - c$. Hence $\langle v, u \rangle \geq 0$ for all $u \in \ker(A)$. But $-u \in \ker(A)$, so $\langle v, u \rangle = 0$ for all $u \in \ker(A)$.

Suppose there is no $y \in \mathbb{R}^p$ with $v = A^*y$. So $v \notin A^*(\mathbb{R}^p) := W$. Since $W$ is nonempty, closed and convex, there exists a nonzero $\pi$ such that
\[
\text{sup}\{ \langle \pi, w \rangle | w \in W \} < \langle \pi, v \rangle.
\]

Since $0 \in W$, so $\langle \pi, v \rangle > 0$.
Also $\langle \pi, A^*(ty) \rangle < \langle \pi, v \rangle$, $\forall t \in \mathbb{R}$, $\forall y \in \mathbb{R}^p$. Therefore $\langle \pi, A^*y \rangle = 0$ (otherwise, we can choose $t$ so that the above inequality doesn’t hold).

Therefore, $\langle A\pi, y \rangle = 0$ for all $y$. So $A\pi = 0$. Hence $\pi \in \ker A$ and $\langle v, \pi \rangle > 0$. This is a contradiction.

Therefore, $N(x; C) \subset \text{im}(A^*)$.

Conversely, suppose $v = A^*y$. For any $x \in C$, we have
\[
\langle v, x - x \rangle = \langle A^*y, x - x \rangle = \langle y, Ax - A\pi \rangle = 0
\]
Hence $v \in N(x; C)$. 

\[\square\]