## ORDER OF GROWTH OF $1/\Gamma$

In this short note, we will establish the order of growth of the entire function $1/\Gamma$.

**Proposition 1.** *There exists constants $A, B \in \mathbb{R}$ such that*

$$\frac{1}{\Gamma(s)} \leq Ae^{B|s|\log|s|}$$

*for all $s \in \mathbb{C}$.*

*Proof.* We knew

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

as meromorphic functions on $\mathbb{C}$, and that will be the key to our proof.

Observe also that it suffices to show the existence of two constants $A, B \in \mathbb{R}$ such that

$$\frac{1}{\Gamma(1-s)} \leq Ae^{B|s|\log|s|} \tag{1}$$

for all $s \in \mathbb{C}$. Indeed then

$$\frac{1}{\Gamma(s)} \leq Ae^{B|1-s|\log|1-s|},$$

which is bounded by a constant if $|s| \leq 2$, and is bounded by $Ae^{B(2|s|)\log(|s|^2)} = Ae^{4B|s|\log|s|}$ if $|s| \geq 2$.

Now to prove (1), we use

$$\frac{1}{\Gamma(1-s)} = \Gamma(s)\frac{\sin(\pi s)}{\pi}.$$

We need to prove that $\Gamma(s)\sin(\pi s)$ is bounded by $Ae^{B|s|\log|s|}$. Note that

$$|\sin \pi s| = \left|\frac{e^{i\pi s} + e^{-i\pi s}}{2i}\right| \leq e^{\pi|s|}, \tag{2}$$

and if $\operatorname{Re} s \geq 1/2$, then

$$|\Gamma(s)| \leq Ae^{|s|\log|s|}. \tag{3}$$

This is because then

$$|\Gamma(s)| \leq \int_0^\infty e^{-t}t^{\operatorname{Re} s-1}dt \leq \int_0^1 e^{-t}t^{\operatorname{Re} s-1}dt + \int_1^\infty e^{-t}t^{\operatorname{Re} s-1}dt.$$

The first integral is bounded by an absolute constant, and if $\sigma$ is the greatest integer smaller than or equal to $\operatorname{Re} s$, then the second integral is bounded by

$$\int_0^\infty e^{-t}t^{(\sigma+1)-1}dt = \Gamma(\sigma+1) = \sigma! \leq \sigma^\sigma = e^{\sigma\log\sigma} \leq Ae^{|s|\log|s|}.$$

Combining (2) and (3), we see that

$$|\Gamma(s)| \leq Ae^{2|s|\log|s|} \quad \text{whenever } \mathrm{Re}\, s \geq 1/2. \tag{4}$$

Now if $|\mathrm{Re}\, s| \leq 1/2$, we consider two cases. If $|\mathrm{Im}\, s| \leq 2$, then $|\Gamma(s)\sin\pi s|$ is bounded by a constant, since $\Gamma(s)\sin\pi s$ defines a continuous function on the compact set $\{s\colon |\mathrm{Re}\, s| \leq 1/2, |\mathrm{Im}\, s| \leq 2\}$. On the other hand, if $|\mathrm{Im}\, s| \geq 2$, then we use the functional equation of $\Gamma$, and the periodicity of $|\sin(\pi s)|$, to get

$$|\Gamma(s)\sin(\pi s)| = \left|\frac{1}{s}\Gamma(s+1)\sin(\pi(s+1))\right| \leq |\Gamma(s+1)\sin(\pi(s+1))|;$$

the point is that $\mathrm{Re}\,(s+1) \geq 1/2$ when $|\mathrm{Re}\, s| \leq 1/2$. Hence by (4), we see that

$$|\Gamma(s)\sin(\pi s)| \leq Ae^{2|s+1|\log|s+1|} \leq Ae^{8|s|\log|s|}$$

if $|\mathrm{Im}\, s| \geq 2$. Together we see that

$$|\Gamma(s)\sin(\pi s)| \leq Ae^{8|s|\log|s|} \quad \text{whenever } |\mathrm{Re}\, s| \leq 1/2. \tag{5}$$

Finally, if $\mathrm{Re}\, s < -1/2$, let $m$ be the positive integer satisfying $-m - \frac{1}{2} \leq \mathrm{Re}\, s < -m + \frac{1}{2}$. By the functional equation of $\Gamma$, and the periodicity of $|\sin(\pi s)|$, we have

$$|\Gamma(s)\sin(\pi s)| = \left|\frac{1}{s(s+1)\dots(s+m-1)}\Gamma(s+m)\sin(\pi(s+m))\right|.$$

Note that $|\mathrm{Re}\,(s+m)| \leq 1/2$, and that

$$|s(s+1)\dots(s+m-1)| \geq |\mathrm{Re}\, s||\mathrm{Re}\, s+1|\dots|\mathrm{Re}\, s+m-1| \geq \frac{1}{2}$$

(all but the last factor are at least 1, and the last factor is at least 1/2). From (5), we have

$$|\Gamma(s)\sin(\pi s)| \leq 2Ae^{8|s+m|\log|s+m|}.$$

Since

$$|s+m| = \left(|\mathrm{Re}\,(s+m)|^2 + |\mathrm{Im}\, s|^2\right)^{1/2} \leq (|\mathrm{Re}\, s|^2 + |\mathrm{Im}\, s|^2)^{1/2} = |s|,$$

this shows

$$|\Gamma(s)\sin(\pi s)| \leq 2Ae^{8|s|\log|s|} \quad \text{whenever } \mathrm{Re}\, s < -1/2. \tag{6}$$

Our bound for $1/\Gamma$ follows from (4), (5) and (6).

□