THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018-2019 semester 1 MATH4060
week 12 tutorial

1 Pre-re-mid-term

1. Suppose f is holomorphic on $\mathbb{C} \setminus \{4, 0, 6, i\}$, and the singularities $4, 0, 6, i$ are *not* removable. Find the radius of convergence of the power series expansion of f centered at the following points:

   (a) 1

   (b) $2 + i$

   (c) $2 - i$

   (d) $9 - 4i$

2. Give an example of a function that has an essential singularity at the origin. Explain how you would calculate the residue of the function at the origin, by

   (a) Laurent expansion; and

   (b) contour integration.

3. Let Ω be an open connected subset of $\mathbb{C}$.

   (a) Suppose f is holomorphic on Ω, and suppose $\int_\gamma f(z)dz = 0$ for any smooth closed curve γ contained in Ω. Show that f has a primitive on Ω, i.e. there exists a holomorphic function F on Ω such that $F' = f$.

   (b) Suppose f is holomorphic on Ω, $f(z) \neq 0$ for all $z \in \Omega$, and suppose $\int_\gamma f'(z)/f(z)dz = 0$ for any smooth closed curve γ contained in Ω. Show that there exists a holomorphic g on Ω such that $e^{g(z)} = f(z)$ for all $z \in \Omega$. Hence show that for any positive integer m, there exists a holomorphic function h on Ω such that $h(z)^m = f(z)$ for all $z \in \Omega$. (g and h are called a logarithm and an m-th root of f on Ω, respectively.)

   (c) Show that the assumption that $\int_\gamma f(z)dz = 0$ for all smooth closed curve $\gamma \subset \Omega$ cannot be removed in part (a). Also, identify a topological condition on Ω, so that the assumption that $\int_\gamma f(z)dz = 0$ for all smooth closed curve $\gamma \subset \Omega$ is satisfied by any holomorphic function f on Ω.

2 Riemann map for simply-connected proper domains with sufficiently smooth boundary

The following is an adaptation of Proposition 27.3 in [1].

Let Ω be a simply-connected proper (i.e. not the whole $\mathbb{C}$) domain in $\mathbb{C}$. Suppose the Laplace's equation with Dirichlet boundary data is solvable on Ω. Then the Riemann map $f : \Omega \to \mathbb{D}$ can be constructed by solving PDEs as follows. Such a construction is amenable to numerical computation.

The main idea is as follows. We would like to construct $\log|f|$, which is harmonic and vanishes on $\partial\Omega$. However, the singularity at $a = f^{-1}(0)$ (or from another perspective, the branch cut) needs to be handled with care. Consider $g = f/(z-a)$, which vanishes nowhere, and hence $\log g$ is well defined and everywhere finite. Then $u = \Re \log|g|$ is a harmonic function and $u|\partial\Omega = -\log|z-a|$, which by the setup, can be solved.

Now, to construct f, fix $a \in \Omega$, and solve for the harmonic function u with boundary value $-\log|z-a|$, as well as its harmonic conjugate v. Define $f(z) = (z-a)\exp(u+iv)$. It remains to show f is the desired conformal map.

f is clearly holomorphic, and by maximum principle, f maps Ω into $\mathbb{D}$. By the defining formula, since exp never vanishes, f has a simple zero at a and no other zeros. Then by argument principle, f attains every point in $\mathbb{D}$ exactly once, and hence is bijective.

The Riemann maps thus constructed for two simple domains are visualized in Figure 1. The triangulation algorithm [2] is used to generate the meshes.

More numerically sophisticated methods for computing conformal maps from surfaces to a planar or simple domain may be found in [3, 4].

References

[1] O. Forster, *Non-compact Riemann Surfaces* in Lectures on Riemann Surfaces, Grad. Texts Math., 81, Springer, New York, 1981, pp. 175-235.

[2] J. R. Shewchuk, *Triangle: Engineering a 2D Quality Mesh Generator and Delaunay Triangulator*, in Applied Computational Geometry: Towards Geometric Engineering, M.C. Lin and D. Manocha, eds., Springer-Verlag, Berlin, 1996, pp. 203-222.

[3] X. Gu, F. Luo and S.T. Yau, *Recent Advances in Computational Conformal Geometry*, Comm. Info. Syst. 9 (2009), pp. 163-196.

[4] B. Springborn, P. Schroder, U. Pinkall, *Conformal Equivalence of Triangle Meshes*, in ACM. Siggraph, 2008.