THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 4050 Real Analysis

Special Tutorial 3 (February 20)

The following were discussed in the tutorial this week.

1. Definition of measurable functions.

2. Example of a non-measurable function f such that $\{x : f(x) = \alpha\}$ is measurable for any $\alpha \in \mathbb{R} \cup \{\pm\infty\}$.

3. A function f is measurable if and only if $f^{-1}(B)$ is measurable for any Borel set B.

4. (a) The composition of two measurable functions may not be measurable.
   (b) Suppose $f : E \to \mathbb{R}$ is a measurable function on a measurable set E and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Then $g \circ f$ is measurable.

5. (HW4 Q4)
   (a) Let D and E be measurable sets and f a function with domain $D \cup E$. Show that f is measurable if and only if its restrictions to D and E are measurable.
   (b) Let f be a function with measurable domain D. Show that f is measurable if and only if the function g defined by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

6. (HW4 Q5)
   (a) Let f be an extended real-valued function with measurable domain D, and let $D_1 = \{x : f(x) = \infty\}$, $D_2 = \{x : f(x) = -\infty\}$. Then f is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable.
   (b) Prove that the product of two measurable extended real-valued function is measurable.

7. Littlewood's three principles.

8. (Borel-Cantelli lemma) Suppose $\{E_n\}_{n=1}^{\infty}$ are measurable sets such that

$$\sum_{n=1}^{\infty} m(E_n) < +\infty.$$

   Then $E := \{x : x \in E_n \text{ for infinitely many } n\}$ is measurable with $m(E) = 0$.

9. (Lusin's Theroem) Suppose f is a finite valued, measurable function on E, with E measurable and $m(E) < \infty$. Then for any $\varepsilon > 0$, there exists a closed set $F_\varepsilon \subseteq E$ with $m(E \setminus F_\varepsilon) < \varepsilon$ and such that $f\big|_{F_\varepsilon}$ is continuous.

10. Let E be a subset of $\mathbb{R}$ with $m^*(E) > 0$. Show that for each $0 < \alpha < 1$, there exists a (finite) open interval I so that

$$m^*(E \cap I) \geq \alpha m^*(I).$$

11. Suppose $E \subseteq \mathbb{R}$ is measurable with $m(E) > 0$. Prove that the difference set of E,

$$E - E := \{x - y \in \mathbb{R} : x, y \in E\}$$

contains an open interval centered at the origin.