THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 4050 Real Analysis

Special Tutorial 2 (February 15)

The following were discussed in the tutorial this week.

1. Construct the Cantor set $\mathcal{C}$ by iteratively removing the open middle-1/3 intervals. Show that the Cantor set is uncountable, and has measure zero.

2. Construct the Cantor function $g : [0, 1] \to [0, 1]$. Show that it is continuous, increasing and constant on each removed intervals, and maps the Cantor set onto $[0, 1]$.

3. Show that every measurable subset of $\mathbb{R}$ with positive measure contains a non-measurable set.

4. Let $\varphi : [0, 1] \to \mathbb{R}$ be defined by $\varphi(x) = x + g(x)$, where g is the Cantor function. Then φ is continuous, strictly increasing, and maps $[0, 1]$ onto $[0, 2]$. Moreover, we show that $\varphi(\mathcal{C})$ is a Borel set with $m(\varphi(\mathcal{C})) = 1$.

5. Using the results in 3 and 4, we prove that

   (a) A non-measurable set can be mapped onto a measurable set by a homeomorphism, and

   (b) the Borel σ-algebra is strictly smaller than the σ-algebra of all (Lebesgue) measurable sets.

6. A brief solution of HW4 Q2.