THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 4050 Real Analysis

Tutorial 3 (March 8)

The following were discussed in the tutorial this week.

1. (a) Fatou's Lemma.
   (b) The inequality can be strict.
   (c) Nonnegativity of the sequence of functions cannot be dropped.
   (d) Generalization: Let $\langle f_n \rangle$ be a sequence of nonnegative measurable functions on E. Then
   $$\int_E \liminf_n f_n \leq \liminf_n \int_E f_n.$$
2. (a) Monotone Convergence Theorem.
   (b) It need not hold for negative or decreasing sequence of functions.
   (c) It is true for decreasing sequence if we further assume that $\int f_1 < +\infty$.
3. Let f be a nonnegative integrable function on $\mathbb{R}$. Show that the function F defined by
   $$F(x) = \int_{-\infty}^{x} f$$
   is continuous.
4. Let f be an integrable function on $[0, 1]$. Show that there exists $c \in [0, 1]$ such that $\int_0^c f = \int_c^1 f$.
5. (a) Lebesgue's Dominated Convergence Theorem
   (b) The domination condition cannot be dropped.
6. The improper Riemann integral of a function may exists without the function being integrable (in the sense of Lebesgue), e.g., $f(x) = \sin x/x$ on $[0, \infty)$. If f is Riemann integrable on $[0, N]$ for all $N \in \mathbb{N}$ and is (Lebesgue) integrable on $[0, \infty)$, then its improper Riemann integral is equal to its Lebesgue integral.
7. Evaluate $\displaystyle\lim_{n\to\infty} \int_0^\infty n^2 e^{-nx} \tan^{-1} x \, dx$.