Suggested Solution of Assignment 9

1. Let $f : \mathbb{R} \to [0, \infty)$ be measurable. By the 2nd principle of Littlewood (one of its version, see Q4 of HW7) there exists a montonically increasing sequence φ_n of non-negative simple functions vanishing outside $(-n, n)$ convergent a.e. to f. Show that, if f is also integrable, then

$$\lim_n \int \varphi_n = \int f \quad \text{and} \quad \lim_n \int \varphi_n(x+c)\,dx = \int f(x+c)\,dx \quad \text{for all } c \in \mathbb{R}.$$

Show further that

$$\int f(x+c)\,dx = \int f(x)\,dx, \quad \forall\, c \in \mathbb{R},$$

and

$$\int f(\lambda x)\,dx = \frac{1}{|\lambda|} \int f(y)\,dy, \quad \forall\, \lambda \neq 0.$$

Solution. See ThA2 and ThA3 in Chapter 7 of lecture notes. ◀

2. A subset Z of a linear space Y with a semi-norm ($\|y\| \geq 0 \forall\, y \in Y$ such that $\|\lambda y\| = |\lambda| \cdot \|y\|$ and $\|y_1 + y_2\| \leq \|y_1\| + \|y_2\|$ $\forall\, \lambda \in \mathbb{R}$, $\forall\, y, y_1, y_2 \in Y$) is said to be dense if for each y in Y and each positive r there exists $z \in Z$ such that $\|y - z\| < r$. Show that each of the following subclasses is dense in $L(\mathbb{R})$ with respect to the semi-norm $\|f\| := \int |f|$.

$$\begin{aligned}
\mathcal{S}_{00}(\mathbb{R}) &:= \{f : \text{ simple functions vanishing outside a finite interval } \}, \\
\mathcal{S}_{t0}(\mathbb{R}) &:= \{f : \text{ step functions vanishing outside a finite interval } \}, \\
\mathcal{C}_{00}(\mathbb{R}) &:= \{f : \text{ continuous functions vanishing outside a finite interval } \}.
\end{aligned}$$

(**Hint:** since each of the subclasses is stable respect to lattice-operations, you need only show that each non-negative f from $L(\mathbb{R})$ can be approximated by non-negative elements from the subclasses.)

Solution. See Theorem 1, 2 and 3 in Chapter 7 of lecture notes. ◀

3. Try some from a subclass and make use of Q1,2 above or Littlewood's principles, show the following results. Let f be an integrable function on $\mathbb{R}$.

   (i) Let a_n, b_n be the "Fourier coefficients" of f:

   $$a_n := \int f(x) \sin nx\,dx, \qquad b_n := \int f(x) \cos nx\,dx \qquad (n \in \mathbb{N}).$$

   Show that $\lim_n a_n = 0 = \lim b_n = 0$.

   (ii) $\lim_{\delta \to 0} \int |f(x) - f(x+\delta)|\,dx = 0$. (**Hint:** each $f \in \mathcal{C}_{00}(\mathbb{R})$ is uniformly continuous.)

Solution. See ThA1 in Chapter 7 of lecture notes. ◀

4. Let f be a function of two variables (x, t) which is defined on the product $Q = [a, b] \times [c, d]$ of intervals such that for each t, the function is measurable on $[a, b]$. Show that:

(i) Suppose $g \in L[a,b]$ such that $|f(x,t)| \leq g(x)\ \forall\,(x,t) \in Q$. Then, $\forall t_0 \in [c,d]$,

$$\lim_{t\to t_0} \int_a^b f(x,t)\,dx = \int_a^b \left(\lim_{t\to t_0} f(x,t)\right)\,dx,$$

provided that, $\forall\, x \in [a,b]$, $\lim_{t\to t_0} f(x,t)$ exists (**Hint:** For $\Phi : \mathbb{R} \to \mathbb{R}$, $t_0 \in \mathbb{R}$, $\lim_{t\to t_0} \Phi(t)$ exists if and only if $\lim_n \Phi(t_n)$ exists whenever (t_n) is a sequence converging to t_0.

(ii)

$$\frac{d}{dt}\int_c^d f(x,t)\,dx = \int_c^d \frac{\partial f(x,t)}{\partial t}\,dx$$

provided that

1) $\dfrac{\partial f}{\partial t}(x,t)$ exists in $\mathbb{R}$, $\forall x,t$;

2) $\left|\dfrac{\partial f}{\partial t}(x,t)\right| \leq G(x)$ on Q, where $G \in L[a,b]$.

Hint: Let $t_0 \in [c,d]$ and $t_n \to t_0$ $(t_n \neq t_0)$. Let $F_n(x) := \dfrac{f(x,t_n) - f(x,t_0)}{t_n - t_0} = f(x,\overline{t_n}(x))$, by Mean Value Theorem, where $\overline{t_n}(x)$ lies between t_0 and t_n. Then $|F_n(x)| \leq G(x)\,\forall\,x$ (and also $\lim_n F_n(x) = \dfrac{\partial f(x,t_0)}{\partial t}$). Hence $\int_c^d F_n(x)\,dx \to \int_c^d \dfrac{\partial f(x,t_0)}{\partial t}\,dx$.

Solution. See ThA4 in Chapter 7 of lecture notes. ◀

5. Let $F \in \mathrm{BV}[0,1] \cap \mathcal{C}[0,1]$ and be ABC in the interval $[a,1]$ for each a with $0 < a \leq 1$. Show that f is ABC on $[0,1]$. (**Hint:** Use the continuity of the indefinite integral defined by F', and also use the fundamental theorem of calculus applied to F. And finally pass to the limit as (F is continuous at 0).

Solution. Since $F \in \mathrm{BV}[0,1]$, F' exists a.e. and $F' \in \mathcal{L}[0,1]$. Let $x \in (0,1]$ and $n \in \mathbb{N}$. Since $F \in \mathrm{ABC}[1/n,1]$, it follows from the Fundamental Theorem of Calculus that

$$F(x) - F(1/n) = \int_{1/n}^x F' \qquad \text{for all sufficiently large } n.$$

Note that $|F'\chi_{[1/n,x]}| \leq |F'|$ on $[0,1]$ and $F' \in \mathcal{L}[0,1]$. Hence, by Dominated Convergence Theorem, $\lim_{n\to\infty} \int_{1/n}^x F' = \int_0^x F'$. As F is continuous at 0, we have

$$F(x) - F(0) = \int_0^x F' \qquad \text{for all } x \in [0,1].$$

Now the second part of the Fundamental Theorem of Calculus yields that $F \in \mathrm{ABC}[0,1]$. ◀

6. Show that $\mathrm{ABC}[a,b]$ is stable with respect to linear operations and multiplication (also quotient f/g if g is bounded away from zero by a positive constant). Show the validity of “integration by parts”.

Solution. It suffices to show that ABC$[a,b]$ is stable under multiplication. Suppose $f,g\in$ ABC$[a,b]$. Then f,g are continuous on $[a,b]$ and there exist $M,N>0$ such that $|f(x)|\le M$ and $|g(x)|\le N$ for all $x\in[a,b]$. Let $\varepsilon>0$ be given. Choose $\delta>0$ to be the constant that corresponds to $\varepsilon/(M+N)$ in the definition of $f,g\in$ ABC$[a,b]$. Now, if $\{(x_i,y_i)\}_{i=1}^n$ is a finite collection of non-overlapping intervals in $[a,b]$ such that $\sum_{i=1}^n|x_i-y_i|<\delta$, then

$$\begin{aligned}\sum_{i=1}^n|f(x_i)g(x_i)-f(y_i)g(y_i)| &\le \sum_{i=1}^n\left(|f(x_i)||g(x_i)-g(y_i)|+|g(y_i)||f(x_i)-f(y_i)|\right)\\ &\le M\sum_{i=1}^n|g(x_i)-g(y_i)|+N\sum_{i=1}^n|f(x_i)-f(y_i)|\\ &< M\cdot\frac{\varepsilon}{M+N}+N\cdot\frac{\varepsilon}{M+N}=\varepsilon.\end{aligned}$$

Hence $fg\in$ ABC$[a,b]$.

Next we show the validity of "integration by parts". Suppose $f,g\in$ ABC$[a,b]$. Then $fg\in$ ABC$[a,b]$ by above. In particular, $f',g',(fg)'$ exist a.e. and $f',g',(fg)'\in\mathcal{L}[a,b]$. By the product rule of differentiation, we have

$$(fg)'(x)=f'(x)g(x)+f(x)g'(x)\qquad\text{for a.e. } x\in[a,b].$$

Before we take the integration, we need to check that the integrands are integrable. Indeed, $f'g,fg'\in\mathcal{L}[a,b]$ since

$$\int|f'g|\le N\int|f'|<\infty\quad\text{and}\quad\int|fg'|\le M\int|g'|<\infty.$$

Now, by the Fundamental Theorem of Calculus,

$$\int_a^b f'g+\int_a^b fg'=\int_a^b(fg)'=(fg)\Big|_a^b.$$

That is

$$\int_a^b f'g=(fg)\Big|_a^b-\int_a^b fg'.$$

◀

7. (Two runners' lemma). Let f,g be integrable on $[a,b]$ such that $\int_a^x f=\int_a^x g$ for each $x\in[a,b]$. Show that $f=g$ a.e.

Solution. Without loss of generality, we can assume that $g\equiv 0$.

Let $G\subseteq(a,b)$ be an open set. By the structure theorem, $G=\bigcup_{n=1}^\infty I_n$, where $\{I_n\}_{n=1}^\infty$ is a countable collection of disjoint open intervals. Write $I_n=(a_n,b_n)$. Then, for $N\in\mathbb{N}$,

$$\int f\chi_{\bigcup_{n=1}^N I_n}=\int f\left(\sum_{n=1}^N\chi_{I_n}\right)=\sum_{n=1}^N\int_{I_n}f=\sum_{n=1}^N\left(\int_a^{b_n}f-\int_a^{a_n}f\right)=0.$$

Note that $|f\chi_{\bigcup_{n=1}^N I_n}\le|f|$, $|f|\in\mathcal{L}[a,b]$ and $\lim_{N\to\infty}f\chi_{\bigcup_{n=1}^N I_n}=f\chi_G$. Hence, by Dominated Convergence Theorem,

$$\int_G f=\lim_{N\to\infty}\int f\chi_{\bigcup_{n=1}^N I_n}=0.$$

Let $B \subseteq (a,b)$ be a closed set. Then $(a,b) \setminus B$ is open. Hence

$$\int_B f = \int_a^b f - \int_{(a,b)\setminus B} f = 0 - 0 = 0.$$

For each $n \in \mathbb{N}$, let $C_n = \{x \in (a,b) : f(x) > 1/n\}$. Let F_n be a closed set such that $F_n \subseteq C_n$ and $m(C_n \setminus F_n) < 1/n$. By above,

$$0 = \int_{F_n} f \geq \int_{F_n} \frac{1}{n} = \frac{1}{n} \cdot m(F_n),$$

so that $m(F_n) = 0$. Hence $m(C_n) \leq m(F_n) + m(C_n \setminus F_n) < 1/n$. Since C_n is increasing, we have

$$m(\{x \in (a,b) : f(x) > 0\}) = m(\bigcup_{n=1}^{\infty} C_n) = \lim_{n\to\infty} m(C_n) = 0,$$

Similarly $m(\{x \in (a,b) : f(x) < 0\}) = 0$. Therefore $f = 0$ a.e. on (a,b), and thus $f = 0$ a.e. on $[a,b]$. ◀