Suggested Solution of Assignment 8

Let $f : [a, b] \to \mathbb{R}$ and $\pi = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \text{par}[a, b]$, a partition of $[a, b]$. Define

$$t(f; \pi) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

and

$$T_a^b(f) := \sup\{t(f; \pi) : \pi \in \text{par}[a, b]\} \ (\leq +\infty).$$

1. Show that $t(f; \pi) \uparrow_\pi$: if partitions $\pi \subseteq \pi'$ then $0 \leq t(f; \pi) \leq t(f; \pi')$.

   Solution. It suffices to consider the simplest case that π' is obtained from π by adding one partition point. Suppose $\pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and $\pi' = \{a = x_0 < x_1 < \cdots < x_k < z < x_{k+1} < \cdots < x_n = b\}$. Then

$$\begin{aligned} 0 &\leq t(f; \pi) \\ &= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &\leq \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + |f(x_k) - f(z)| + |f(z) - f(x_{k+1})| + \sum_{i=k+1}^{n} |f(x_i) - f(x_{i-1})| \\ &= t(f; \pi'). \end{aligned}$$

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2. Show that $T_a^b(f) = T_a^c(f) + T_c^b(f)\ \forall\, c \in (a, b)$ (so $T_a^c(f) \uparrow_c$).

   Solution. Let $\pi = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \text{par}[a, b]$. Suppose $x_k \leq c \leq x_{k+1}$. Then

$$\begin{aligned} t(f; \pi) &= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &\leq \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + |f(x_k) - f(c)| + |f(c) - f(x_{k+1})| + \sum_{i=k+1}^{n} |f(x_i) - f(x_{i-1})| \\ &\leq T_a^c(f) + T_c^b(f), \end{aligned}$$

   so that $T_a^b(f) \leq T_a^c(f) + T_c^b(f)$. On the other hand, let $\pi_1 \in \text{par}[a, c]$ and $\pi_2 \in \text{par}[c, b]$, then $\pi_1 \cup \pi_2 \in \text{par}[a, b]$ and

$$T_a^b(f) \geq t(f; \pi_1 \cup \pi_2) = t(f; \pi_1) + t(f; \pi_2).$$

   Consequently $T_a^b(f) \geq T_a^c(f) + T_c^b(f)$. ◀

3. Do Q1, Q2 similarly for $p(t; \pi) = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+$ and $P_a^b(f) := \sup\{p(f; \pi) : \pi \in \text{par}[a, b]\}$. Also for $n(t; \pi)$ and $N_a^b(f)$ (with $[f(x_i) - f(x_{i-1})]^-$ in place of $[f(x_i) - f(x_{i-1})]^+$).

   Solution. Follow the same argument as in Q1 and Q2 together with the fact that

$$(a + b)^+ \leq a^+ + b^+ \quad \text{and} \quad (a + b)^- \leq a^- + b^-.$$

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4. Show $T_a^b(f) = P_a^b(f) + N_a^b(f)$. Hint: Let $\pi_1, \pi_2 \in \text{par}[a, b]$. Let $\pi_1 \cup \pi_2 \in \text{par}[a, b]$ consisting of all partition points of π_1 and π_2. Then, since $|a| = a^+ + a^-\ \forall\, a \in \mathbb{R}$,

$$T_a^b(f) \geq t(f; \pi_1 \cup \pi_2) = p(f; \pi_1 \cup \pi_2) + n(f; \pi_1 \cup \pi_2) \geq p(f; \pi_1) + n(f; \pi_2),$$

since this is true $\forall\, \pi_1, \pi_2 \in \text{par}[a, b]$, it follows that $T_a^b(f) \geq P_a^b(f) + N_a^b(f)$. The opposite inequality is easy.

Solution. It is clear from the hint. ◀

5. Show that $f(b) - f(a) = P_a^b(f) - N_a^b(f)$ if $N_a^b(f) \in \mathbb{R}$

Solution. For any partition $\pi = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \text{par}[a, b]$,

$$\sum_{i=1}^n [f(x_i) - f(x_{i-1})]^+ - \sum_{i=1}^n [f(x_i) - f(x_{i-1})]^- = \sum_{i=1}^n [f(x_i) - f(x_{i-1})] = f(b) - f(a),$$

so that

$$\sum_{i=1}^n [f(x_i) - f(x_{i-1})]^+ \leq N_a^b(f) + f(b) - f(a),$$

and consequently $P_a^b(f) \leq N_a^b(f) + f(b) - f(a)$. Similarly, $N_a^b(f) \leq P_a^b(f) + f(b) - f(a)$. Hence $f(b) - f(a) = P_a^b(f) - N_a^b(f)$ whenever $N_a^b(f) \in \mathbb{R}$. ◀

6. Let $f \in \text{BV}[a, b]$, that is $T_a^b(f) < +\infty$. Then $P_a^x(f), N_a^x(f)$ $(x \in [a, b])$ are $\uparrow$-functions from $[a, b]$ into $\mathbb{R}$ and $f(x) = f(a) + P_a^x(f) - N_a^x(f)\ \forall x \in [a, b]$.

Solution. By Q3, $P_a^x(f)$ is increasing in $x \in [a, b]$. Moreover,

$$P_a^x(f) \leq P_a^b \leq T_a^b < +\infty \quad \text{for all } x \in [a, b].$$

Hence $P_a^x(f)$ is a real-valued function. Similarly, one can show that $N_a^x(f)$ is an increasing function from $[a, b]$ into $\mathbb{R}$. ◀

7. Let $f \in \text{ABC}[a, b]$. Show that $f \in \text{BV}[a, b]$: Let $\varepsilon := 1$, and take $\delta > 0$ accordingly in the definition of absolute continuity of f. Take $N \in \mathbb{N}$ such that $\dfrac{b-a}{N} < \delta$ and divide $[a, b]$ into N-many subintervals of equal length with partition points

$$a = x_0 < x_1 < x_2 \cdots < x_{N-1} < x_N = b.$$

Show that $T_{x_{i-1}}^{x_i}(f) \leq \varepsilon = 1$ and so $T_a^b(f) \leq N$.

Solution. Following the hint above, we let $\pi' = \{x_{i-1} = y_0 < y_1 < \cdots y_n = x_i\}$ be a partition of $[x_{i-1}, x_i]$. Then $\displaystyle\sum_{j=1}^n |y_j - y_{j-1}| = x_i - x_{i-1} < \delta$. It follows from the absolute continuity of f that

$$t(f; \pi') = \sum_{j=1}^n |f(y_j) - f(y_{j-1})| < \varepsilon,$$

so that $T_{x_{i-1}}^{x_i}(f) \leq \varepsilon = 1$. Finally $T_a^b(f) = \sum_{i=1}^N T_{x_{i-1}}^{x_i}(f) \leq N$, and so $f \in \text{BV}[a, b]$. ◀

8. Let $0 \le f \in \mathcal{L}[a,b]$ and let $F(x) = \displaystyle\int_a^x f \ \forall x \in [a,b]$. Show that $F \in \mathrm{ABC}[a,b]$. Can you drop the condition $f \ge 0$? (Yes as $f = f^+ - f^-$; also $F_1, F_2 \in \mathrm{ABC}[a,b] \implies F_1 \pm F_2 \in \mathrm{ABC}[a,b]$.)

   Solution. We only prove the case where $f \ge 0$. Let $\varepsilon > 0$. Since $0 \le f \in \mathcal{L}[a,b]$, there exists $\delta > 0$ such that

$$\int_A f < \varepsilon \quad \text{whenever } A \subseteq [a,b] \text{ with } m(A) < \delta.$$

   Now if $\{(x_i, x_i')\}_{i=1}^n$ is a finite collection of non-overlapping intervals with $\sum_{i=1}^n |x_i' - x_i| < \delta$, then $m(\bigcup_{i=1}^n (x_i, x_i')) < \delta$, so that

$$\sum_{i=1}^n |F(x_i') - F(x_i)| = \sum_{i=1}^n \int_{x_i}^{x_i'} f = \int_{\bigcup_{i=1}^n (x_i, x_i')} f < \varepsilon.$$

   Hence $f \in \mathrm{ABC}[a,b]$. ◀

9. Let $f : [a,b] \to \mathbb{R}$ be $\uparrow$ (or $\downarrow$). Show that $f^{-1}(I)$ is measurable whenever I is an interval and hence f is measurable. (Hint: use the characteristic property for an interval: order convexity.)

   Solution. Assume that f is increasing. Let I be an interval. If $f^{-1}(I)$ is an emtpy set or a singleton, then it is clearly measurable. On the other hand, suppose $x, y \in f^{-1}(I)$ and $x < y$. If $x < z < y$, then $f(x) \le f(z) \le f(y)$ since f is increasing. Using the characteristic property of interval on I, we have $f(z) \in I$, that is $z \in f^{-1}(I)$. By the characteristic property of interval again, we conclude that $f^{-1}(I)$ is an interval. ◀