Suggested Solution of Assignment 1

1.* (3rd: P.12, Q6)

Let $f : X \to Y$ be a mapping of a nonempty space X into Y. Show that f is one-to-one if and only if there is a mapping $g : Y \to X$ such that $g \circ f$ is the identity map on X, that is, such that $g(f(x)) = x$ for all $x \in X$.

Solution. Suppose f is one-to-one. Thus, for each $y \in f[X]$, there exists a unique $x_y \in X$ such that $f(x_y) = y$. Fix $x_0 \in X$. Define $g : Y \to X$ by

$$g(y) = \begin{cases} x_y & \text{if } y \in f[X], \\ x_0 & \text{if } y \in Y \setminus f[x]. \end{cases}$$

Then g is a well-defined mapping and $g \circ f$ is the identity map on X.

On the other hand, suppose that such mapping g exists. If $f(x_1) = f(x_2)$, $x_1, x_2 \in X$, then

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2.$$

Hence f is one-to-one. ◀

2. (3rd: P.12, Q7)

Let $f : X \to Y$ be a mapping of X into Y. Show that f is onto if there is a mapping $g : Y \to X$ such that $f \circ g$ is the identity map in Y, that is, $f(g(y)) = y$ for all $y \in Y$.

Solution. Suppose f is onto. For each $y \in Y$, there exists $x_y \in X$ such that $f(x_y) = y$. Define $g : Y \to X$ by $g(y) = x_y$. Then g is a well-defined mapping and $f \circ g$ is the identity map on Y.

Conversely, suppose that such mapping g exists. For any $y \in Y$, $x := g(y) \in X$ satisfies

$$f(x) = f(g(y)) = y.$$

Hence f is onto. ◀

3. Show that any set X can be "indexed": $\exists$ a set I and a function $f : I \to X$ such that $\{f(i) : i \in I\} = X$.

Solution. Simply take $I = X$ and $f : I \to X$ to be the identity function. ◀

4.* (3rd: P.16, Q14)

Given a set B and a collection of sets $\mathcal{C}$. Show that

$$B \cap \left[\bigcup_{A \in \mathcal{C}} A \right] = \bigcup_{A \in \mathcal{C}} (B \cap A).$$

Solution.

$$\begin{aligned} x \in B \cap \left[\bigcup_{A \in \mathcal{C}} A \right] &\iff x \in B \text{ and } x \in \bigcup_{A \in \mathcal{C}} A \\ &\iff x \in B \text{ and } (x \in A \text{ for some } A \in \mathcal{C}) \\ &\iff x \in A \cap B \text{ for some } A \in \mathcal{C} \\ &\iff x \in \bigcup_{A \in \mathcal{C}} (B \cap A). \end{aligned}$$

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5. (3rd: P.16, Q15)

   Show that if $\mathcal{A}$ and $\mathcal{B}$ are two collections of sets, then

$$[\bigcup\{A : A \in \mathcal{A}\}] \cap [\bigcup\{B : B \in \mathcal{B}\}] = \bigcup\{A \cap B : (A, B) \in \mathcal{A} \times \mathcal{B}\}.$$

   Solution. Using the result in Q4 twice, we have

$$[\bigcup\{A : A \in \mathcal{A}\}] \cap [\bigcup\{B : B \in \mathcal{B}\}] = \bigcup_{B \in \mathcal{B}} [\bigcup\{A : A \in \mathcal{A}\}] \cap B$$

$$= \bigcup_{B \in \mathcal{B}} \left[\bigcup_{A \in \mathcal{A}} (A \cap B)\right] = \bigcup_{(A,B) \in \mathcal{A} \times \mathcal{B}} (A \cap B) = \bigcup\{A \cap B : (A, B) \in \mathcal{A} \times \mathcal{B}\}.$$

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6. (3rd: P.16, Q16)

   Let $f : X \to Y$ be a function and $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of X.

   (a) Show that $f[\bigcup A_\lambda] = \bigcup f[A_\lambda]$.
   (b) Show that $f[\bigcap A_\lambda] \subset \bigcap f[A_\lambda]$.
   (c) Give an example where $f[\bigcap A_\lambda] \neq \bigcap f[A_\lambda]$.

   Solution. (a) If $x \in \bigcup A_\lambda$, then $x \in A_{\lambda_0}$ for some λ_0, so that $f(x) \in f\left[A_{\lambda_0}\right] \subset \bigcup f[A_\lambda]$. Hence $f\left[\bigcup A_\lambda\right] \subset \bigcup f[A_\lambda]$.

   Conversely, if $y \in \bigcup f[A_\lambda]$, then $y \in f[A_{\lambda_0}]$ for some λ_0, so that $y \in f\left[\bigcup A_\lambda\right]$. Hence $\bigcup f[A_\lambda] \subset f\left[\bigcup A_\lambda\right]$.

   (b) If $x \in \bigcap A_\lambda$, then $x \in A_\lambda$ for all λ, so that $f(x) \in f[A_\lambda]$ for all λ. Hence $f(x) \in \bigcap f[A_\lambda]$ and thus $\bigcap f[A_\lambda] \subset f\left[\bigcap A_\lambda\right]$.

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let $A = (-\infty, 0)$ and $B = (0, \infty)$. Then $f(A \cap B) = f(\emptyset) = \emptyset$ while $f(A) \cap f(B) = (0, \infty) \cap (0, \infty) = (0, \infty)$.

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7.* (3rd: P.16, Q17)

   Let $f : X \to Y$ be a function and $\{B_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of Y.

   (a) Show that $f^{-1}[\bigcup B_\lambda] = \bigcup f^{-1}[B_\lambda]$.
   (b) Show that $f^{-1}[\bigcap B_\lambda] = \bigcap f^{-1}[B_\lambda]$.
   (c) Show that $f^{-1}[B^c] = (f^{-1}[B])^c$ for $B \subset Y$.

   Solution. (a)

$$\begin{aligned} x \in f^{-1}\left[\bigcup B_\lambda\right] &\iff f(x) \in \bigcup B_\lambda \\ &\iff (\exists\, \lambda)(f(x) \in B_\lambda) \\ &\iff (\exists\, \lambda)(x \in f^{-1}[B_\lambda]) \\ &\iff x \in \bigcup f^{-1}[B_\lambda]. \end{aligned}$$

(b)

$$\begin{aligned} x \in f^{-1}\left[\bigcap B_\lambda\right] &\iff f(x) \in \bigcap B_\lambda \\ &\iff (\forall\, \lambda)(f(x) \in B_\lambda) \\ &\iff (\forall\, \lambda)(x \in f^{-1}[B_\lambda]) \\ &\iff x \in \bigcap f^{-1}[B_\lambda]. \end{aligned}$$

(c)

$$\begin{aligned} x \in f^{-1}[B^c] &\iff f(x) \in B^c \\ &\iff \neg(f(x) \in B) \\ &\iff \neg(x \in f^{-1}[B]) \\ &\iff x \in (f^{-1}[B])^c. \end{aligned}$$

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8.* (3rd: P.16, Q18)

(a) Show that if f maps X into Y and $A \subset X$, $B \subset Y$, then

$$f[f^{-1}[B]] \subset B$$

and

$$f^{-1}[f[A]] \supset A.$$

(b) Give examples to show that we need not have equality.

(c) Show that if f maps X onto Y and $B \subset Y$, then

$$f[f^{-1}[B]] = B.$$

Solution. (a) It is easy to see that

$$\begin{aligned} y \in f[f^{-1}[B]] &\iff (\exists\, x)(y = f(x) \text{ and } x \in f^{-1}[B]) \\ &\iff (\exists\, x)(y = f(x) \text{ and } f(x) \in B) \\ &\implies y \in B, \end{aligned}$$

and

$$\begin{aligned} x \in A &\implies f(x) \in f[A] \\ &\iff x \in f^{-1}[f[A]]. \end{aligned}$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let $A = [0, \infty)$ and $B = (-\infty, \infty)$. Then

$$f[f^{-1}[B]] = f[(-\infty, \infty)] = [0, \infty) \subsetneq B$$

while

$$f^{-1}[f[A]] = f^{-1}[[0, \infty)] = (-\infty, \infty) \supsetneq A.$$

(c) Suppose f maps X onto Y. Let $y \in B$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. As $y \in B$, we have $x \in f^{-1}[B]$. Hence $y = f(x) \in f[f^{-1}[B]]$. Therefore $f[f^{-1}[B]] \supset B$.

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9. Show that $f \mapsto \int_0^1 f(x)dx$ is a "monotone" function on $\mathcal{R}[0,1]$ (consisting of all Riemann integrable functions on $[0,1]$), and $\mathcal{R}[0,1]$ is a linear space. Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

if $f, f_n \in \mathcal{R}[0,1]$ such that

$$\lim_{n\to\infty} \Big(\sup_{x\in[0,1]} |f_n(x) - f(x)| \Big) = 0.$$

Solution. Recall that $f : [0,1] \to \mathbb{R}$ is Riemann integrable if and only if

$$\lim_{\|P\|\to 0} U(f,P) = \lim_{\|P\|\to 0} u(f,P), \tag{1}$$

where $U(f;P)$ and $u(f;P)$ denote the upper and lower Riemann sum of f, respectively, with respect to a partition P. In this case, $\int_0^1 f(x)dx$ is defined as the common value in (1).

Now suppose $f, g \in \mathcal{R}[0,1]$ and $c \in \mathbb{R}$. Then it is easy to see that

$$u(f;P) + u(g;P) \le u(f+g;P) \le U(f+g;P) \le U(f;P) + U(g;P), \tag{2}$$

which, together with (1), implies that $f + g \in \mathcal{R}[0,1]$ and

$$\int_0^1 (f+g)(x)dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx.$$

With similar arguments, one can show that $cf \in \mathcal{R}[0,1]$ with

$$\int_0^1 cf(x)dx = c\int_0^1 f(x)dx,$$

and

$$\int_0^1 f(x)dx \le \int_0^1 g(x)dx \qquad \text{if } f(x) \le g(x) \text{ on } [0,1].$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that whenever $n \ge N$,

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| < \varepsilon,$$

that is,

$$f(x) - \varepsilon \le f_n(x) \le f(x) + \varepsilon \qquad \text{for all } x \in [0,1].$$

By the monotonicity and linearity of Riemann integral, we have, for all $n \ge N$,

$$\int_0^1 f(x)dx - \varepsilon = \int_0^1 (f(x) - \varepsilon)dx \le \int_0^1 f_n(x)dx \le \int_0^1 (f(x) + \varepsilon)dx = \int_0^1 f(x)dx + \varepsilon,$$

so that

$$\left| \int_0^1 f_n(x)dx - \int_0^1 f(x)dx \right| \le \varepsilon.$$

Therefore

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx.$$

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