$\mathcal{S} = \mathcal{S}(\mathbb{R}) \ni \varphi := \sum_{i=1}^{n} a_i \chi_{E_i}$ with each $a_i \in \mathbb{R}$, $E_i \in \mathfrak{M}$

$\mathcal{S}_0 = \mathcal{S}_0(\mathbb{R}) = \{\varphi \in \mathcal{S} : f = 0 \text{ outside a set of finite mea}\}$

$\mathcal{S}(E) = \{\varphi \in \mathcal{S} : \varphi = 0 \text{ on } \mathbb{R} \setminus E\}$

$\mathcal{S}_0(E) = \{\varphi \in \mathcal{S}_0 : \varphi = 0 \text{ on } \mathbb{R} \setminus E\}$ $\quad \forall E \in \mathfrak{M}$

We have defined $\int_{\mathbb{R}} \varphi \quad \forall \varphi \in \mathcal{S}_0$ and established properties about this integral.

Also $\int_E \varphi \overset{\text{def}}{=} \int_{\mathbb{R}} \varphi \chi_E = \sum_{i=0}^{N} b_i \, m(E \cap \varphi^{-1}(b_i)) \quad \forall \varphi$

where $\{b_1, \dots b_N\}$ is the set of non-zero values of φ and $b_i \neq b_j \ \forall i \neq j$; $b_0 := 0$ (canonical rep of $\varphi := \sum_{i=0}^{N} b_i \chi_{\varphi^{-1}(b_i)}$)

Prop 1. Let $\varphi = \sum_{i=0}^{N} b_i \chi_{B_i} \in \mathcal{S}_0$, $B_i := \varphi^{-1}(b_i)$ as above. Then, $\forall A, E \in \mathfrak{M}$ with $A \subseteq E$,

(i) $\int_A \varphi = \int_{\mathbb{R}} \varphi \chi_A = \int_E (\varphi \chi_A)$

(ii) $\int_{E_1 \cup_o E_2} \varphi = \int_{E_1} \varphi + \int_{E_2} \varphi$, $\forall E_1, E_2 \in \mathfrak{M}$ with $E_1 \cap E_2 = \emptyset$

Proof. (i). Evident as $\int_E (\varphi \chi_A) = \int (\varphi \chi_A) \chi_E$ and $\chi_A \chi_E = \chi_A$.

(ii) RHS $= \sum_{i=0}^{N} b_i \, m(E_1 \cap B_i) + \sum_{i=0}^{N} b_i \, m(E_2 \cap B_i)$

$= \sum_{i=0}^{N} b_i \left[m(E_1 \cap B_i) + m(E_2 \cap B_i) \right] = \sum_{i=0}^{N} b_i \left[m((E_1 \cup_o E_2) \cap B_i) \right]$

Prop 1*. Let $m(E) < +\infty$ and $f \in BF(E)$.
Let A, E_1, E_2 be measurable subsets of E
with $E_1 \cap E_2 = \emptyset$. Then

(i) $\overline{\int_A} f = \overline{\int_E} (f\chi_A)$, i.e.

$$\inf\left\{\int_A \varphi : \mathcal{S}(A) \ni \varphi \geq f \text{ on } A\right\} = \inf\left\{\int_E \psi : \mathcal{S}(E) \ni \psi \geq f\chi_A \text{ on } E\right\} \quad (*)$$

(ii) $\overline{\int_{E_1 \cup E_2}} f = \overline{\int_{E_1}} f + \overline{\int_{E_2}} f$.

(also the corr. results for lower-integrals)

proof. (i). Let φ be as in the LHS of $(*)$. Extend φ to $\bar{\varphi} = \begin{cases} \varphi & \text{on } A \\ 0 & \text{on } E \setminus A \text{ (or on } \mathbb{R} \setminus A) \end{cases}$. Then $\mathcal{S}(E) \ni \bar{\varphi} \geq f\chi_A$ on E and so $\int_E \bar{\varphi} \geq \text{RHS}$ & $\int_E \bar{\varphi} = \int_A \varphi$. This implies that LHS $\geq$ RHS of $(*)$.

Conversely, let ψ be as in RHS of $(*)$. Then $\psi \geq f$ on A and $\psi \geq 0$ on $E \setminus A$, hence

$$\int_E \psi = \int_A \psi + \int_{E \setminus A} \psi \geq \int_A \psi + 0 \geq \text{LHS of } (*)$$

This implies that RHS $\geq$ LHS of $(*)$. so $(*)$ holds.

(ii). Let $\mathcal{S}(E_1 \cup_\circ E_2) \ni \psi \geq f$ on $E_1 \cup_\circ E_2$

Then $\mathcal{S}(E_i) \ni \psi_i \geq f$ on E_i where

$\psi_i = \psi \chi_{E_i}$ (i.e. the restriction to E_i of ψ).

By linearity (cf. Prop 1 (ii)),

$$\int_{E_1 \cup_\circ E_2} \psi = \int_{E_1} \psi_1 + \int_{E_2} \psi_2 \geq \overline{\int_{E_1}} f + \overline{\int_{E_2}} f$$

Taking inf over all such ψ, one has

$$\overline{\int_{E_1 \cup E_2}} f \geq \overline{\int_{E_1}} f + \overline{\int_{E_2}} f \qquad (\#)$$

Conversely, let $\varepsilon > 0$ and then $\exists$
$\psi_i \in \mathcal{S}(E_i)$ such that $\psi_i \geq f$ on E_i &

$$\int_{E_i} \psi_i < \overline{\int_{E_i}} f + \varepsilon/2 \qquad (i = 1, 2)$$

Let $\psi := \psi_1 + \psi_2$ (i.e. $\psi = \begin{cases} \psi_1 & \text{on } E_1 \\ \psi_2 & \text{on } E_2 \end{cases}$).

Then $\psi \in \mathcal{S}(E_1 \cup_\circ E_2)$ with $\psi \geq f$ on $E_1 \cup_\circ E_2$ s.t.

$$\overline{\int_{E_1 \cup_\circ E_2}} f \leq \int_{E_1 \cup_\circ E_2} \psi = \int_{E_1} \psi_1 + \int_{E_2} \psi_2 < \overline{\int_{E_1}} f + \overline{\int_{E_2}} f + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, (ii) is shown (noting (#)).

Questions. Prove/Disprove:

Let $f, g \in BF(E)$ with $m(E) < +\infty$

(i) If $f \leq g$ a.e on E then

$$\overline{\int_E} f \leq \overline{\int_E} g \quad \& \quad \underline{\int_E} f \leq \underline{\int_E} g$$

(ii) If $0 \leq f$ a.e on E and A is a measurable subset of E then

$$0 \leq \underline{\int_A} f \leq \overline{\int_E} f$$