Hw 7

1. Using the Structure Theorem for Open Sets (and check by epsiron-delta terminology) show that each continuous function f on a closed set F in R can be continuously extended to be on the whole of R (this is known as the Tietze extension Theorem).

2. Let f be a measurable real-valued function on a set E of finite measure. Show that there exists a sequence of continuous functions convergent to f almost everywhere on E. Hence, for any r > 0, there exist a closed set F contained in E with m (E\F) < r such that the above convergence is uniform on F and the restriction of f to F is continuous.

3. Let f be a measurable real-valued function on a measurable set E of possibly infinite measure, and let r > 0. Apply Q2 to get a corresponding closed set F_n contained in the intersection of E with (n, n+1] for each integer n. Show that the union F of F_n is closed and that the restriction of f to F is continuous. Moreover we can arrange in such a way that m(E\F) < r.

4. Let f be a non-negative extended real function on a measurable set E. Show that the sequence (f_n) of simple functions monotonically increases and converges point-wisely to f, where

$$f_n := \sum_{K=1}^{n \cdot 2^n} \frac{K-1}{2^n} \chi_{B_{n,K}} + n\chi_{A_n}$$

$$A_n = \{ x \in E \cap [-n, n] : n \leq f(x) \}$$

$$B_{n,K} = \{ x \in E \cap [-n, n] : \frac{K-1}{2^n} \leq f(x) < \frac{K}{2^n} \}, \quad K = 1, \cdots, n \cdot 2^n$$

(This result is also known as Littlewood's 2nd principle - one of the versions)

5. Let $m(E) \leq +\infty$, and $\mathcal{L}(E)$ consist of all measurable functions f on E such that $\int_E |f| < +\infty$. Let $f_n, g_n, f, g \in \mathcal{L}(E)$ be such that $f_n \to f$, $g_n \to g$ and $|f_n| \leq g_n \ \forall n$ (all are ptwise or a.e on E). Suppose further that $\lim_n \int_E g_n = \int_E g$. Show that $\lim_n \int_E |f_n - f| = 0$.

6. Let $f_n, f \in \mathcal{L}(E)$ & $f_n \to f$ a.e on E. Suppose $\lim_n \int_E |f_n| = \int_E |f|$. Show that $\lim_n \int_E f_n = \int_E f$.