ORDINARY DIFFERENTIAL EQUATIONS: MIDTERM EXAMINATION

I. (a) Solve the O.D.E. $y'(t) = \frac{3t^2}{3y^2(t)-6}$ with initial condition $y(1) = 0$. (7 pt)

(b) Solve the O.D.E. $(3y + 2xy^2) + (x + 2x^2y)\frac{dy}{dx} = 0$ with initial condition $y(1) = 1$. (Hint: An integrating factor of the O.D.E. is of the form $h(xy)$, for some continuous function h). (8 pts)

(c) Solve the O.D.E. $4y''(t) - 4y'(t) + y(t) = 8e^{\frac{t}{2}}$. (7 pts)

II. Use the method of reduction of order to find the general solution to the following O.D.E.

$$(\sin t)y'' - (\sin t + \cos t)y' + (\cos t)y = 0, \quad 0 < t < \pi,$$

provided that one solution $y = \varphi_1(t)y = e^t$ is given. (8 pts)

Find the general solution to the O.D.E.

$$(t^2 - 2t + 2)y^{(3)}(t) - t^2y^{(2)}(t) + 2ty'(t) - 2y(t) = 0$$

given two of the solutions $y = \varphi_1(t) = e^t$ and $y = \varphi_2(t) = t$. (12 pts)

III. (a) Let $\{\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)\}$ be a linear independent set of n-times continuously differentiable functions on an open interval $I \subset \mathbb{R}$. Show that there exists a set of continuous functions $\{p_{n-1}(t), \ldots, p_1(t), p_0(t)\}$ on I such that

$$\varphi_j^{(n)}(t) + p_{n-1}(t)\varphi_j^{(n-1)}(t) + \cdots + p_1(t)\varphi_j'(t) + p_0(t)\varphi_j(t) = 0$$

on I, for every $j = 1, \ldots, n$.

(10 pts)

(b) Find a second order linear O.D.E that has $\{e^t, \sin t\}$ as a fundamental set. (7 pts)

IV. Let $A = \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix}$.

(a) Find a Jordan decomposition of A. (8 pts)

(b) Find the general solution to the O.D.E. $x'(t) = Ax(t)$. (8 pts)

V. Let $P(t) = \frac{1}{t}\begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}$, $t > 0$.

(a) Check that $\Psi(t) = \begin{pmatrix} 3t^4 & t^2 \\ -t^4 & -t^2 \end{pmatrix}$ is a fundamental matrix of the homogeneous O.D.E. $x'(t) = P(t)x(t)$ for $t > 0$. (5 pts)

(b) Find the general solution of the O.D.E. $x'(t) = P(t)x + f(t)$, $t > 0$, where $f(t) = \begin{pmatrix} 4t^4 \\ 0 \end{pmatrix}$.(10 pts)

VI. Let $x_1 = y$, $x_2 = y'$, $x_3 = y''$, then the third order equation

$$y^{(3)} + p(t)y'' + q(t)y' + r(t)y = 0 \tag{1}$$

corresponds to the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ x_3' &= -r(t)x_1 - q(t)x_2 - p(t)x_3. \end{aligned} \tag{2}$$

Show that if $\{y_1, y_2, y_3\}$ and $\{\phi_1, \phi_2, \phi_3\}$ are fundamental sets of equation (1) and (2), respectively, then

$$W(y_1, y_2, y_3)(t) = c\hat{W}(\phi_1, \phi_2, \phi_3)(t),$$

where c is a non-zero constant and W and $\hat{W}$ denote the Wronskian functions given by $W(y_1, y_2, y_3) = \det \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix}$ and $\hat{W}(\phi_1, \phi_2, \phi_3) = \det(\phi_1, \phi_2, \phi_3)$. (10 pts)