MATH 3270A Tutorial 9

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1 First order linear system with constant coefficients

Example 1. *Solve the following system.*

$$\begin{cases} y_1' &= 2y_1 + y_2 \\ y_2' &= 4y_1 + 2y_2 \end{cases}$$

Solution

Note that the system can be written as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Now, we find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

The characteristic polynomial is given by

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 4$$

which has roots $\lambda_1 = 0$ *and* $\lambda_2 = 4$. *Note that*

$$v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

are corresponding eigenvectors. Hence, the general solutions are given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{0t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 e^{4t} \\ -2C_1 + 2C_2 e^{4t} \end{pmatrix}$$

Example 2. *Solve the following system.*

$$\begin{cases} y_1' &= 7y_1 - 4y_2 \\ y_2' &= 5y_1 - y_2 \end{cases}$$

Solution

Note that the system can be written as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 7 & -4 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Now, we find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{pmatrix} 7 & -4 \\ 5 & -1 \end{pmatrix}$$

The characteristic polynomial is given by

$$\det(A - \lambda I) = \det \begin{pmatrix} 7-\lambda & -4 \\ 5 & -1-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 13$$

which has roots $\lambda_1 = 3 + 2i$ *and* $\lambda_2 = 3 - 2i$. *Note that*

$$v_1 = \begin{pmatrix} 4+2i \\ 5 \end{pmatrix}, v_2 = \begin{pmatrix} 4-2i \\ 5 \end{pmatrix}$$

are corresponding eigenvectors. Hence, the general solutions are given by

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= C_1 \mathrm{Re}\left(e^{(3+2i)t} \begin{pmatrix} 4+2i \\ 5 \end{pmatrix}\right) + C_2 \mathrm{Im}\left(e^{(3+2i)t} \begin{pmatrix} 4+2i \\ 5 \end{pmatrix}\right) \\ &= \begin{pmatrix} C_1 e^{3t}(4\cos 2t - 2\sin 2t) + C_2 e^{3t}(4\sin 2t + 2\cos 2t) \\ 5C_1 e^{3t}\cos 2t + 5C_2 e^{3t}\sin 2t \end{pmatrix} \end{aligned}$$

Example 3. *Solve the following system.*

$$\begin{cases} y_1' &= 9y_1 - 2y_2 \\ y_2' &= 2y_1 + 5y_2 \end{cases}$$

Solution

Note that the system can be written as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 9 & -2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Now, we find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{pmatrix} 9 & -2 \\ 2 & 5 \end{pmatrix}$$

The characteristic polynomial is given by

$$\det(A - \lambda I) = \det \begin{pmatrix} 9-\lambda & -2 \\ 2 & 5-\lambda \end{pmatrix} = \lambda^2 - 14\lambda + 49$$

which has a double root $\lambda = 7$. *Note that*

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is a corresponding eigenvector. Now, we want to find a generalized eigenvector. To do so, we solve for v_2 such that

$$(A - \lambda I)v_2 = v_1$$
$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

One may take

$$v_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

Hence, the general solutions are given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^{7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{7t} \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \right)$$

Example 4. *Solve the following system.*

$$\begin{cases} y_1' &= -y_1 - 18y_2 - 7y_3 \\ y_2' &= y_1 - 13y_2 - 4y_3 \\ y_3' &= -y_1 + 25y_2 + 8y_3 \end{cases}$$

Solution

Note that the system can be written as

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} -1 & -18 & -7 \\ 1 & -13 & -4 \\ -1 & 25 & 8 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Now, we find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{pmatrix} -1 & -18 & -7 \\ 1 & -13 & -4 \\ -1 & 25 & 8 \end{pmatrix}$$

The characteristic polynomial is given by

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & -18 & -7 \\ 1 & -13-\lambda & -4 \\ -1 & 25 & 8-\lambda \end{pmatrix} = -(\lambda + 2)^3$$

which has a triple root $\lambda = -2$. Note that

$$v_1 = \begin{pmatrix} -5 \\ -3 \\ 7 \end{pmatrix}$$

is a corresponding eigenvector and the geometric multiplicity of $\lambda = -2$ is 1. Now, we want to find two generalized eigenvectors. To do so, we solve for v_2 such that

$$(A - \lambda I)v_2 = v_1$$
$$\begin{pmatrix} 1 & -18 & -7 \\ 1 & -11 & -4 \\ -1 & 25 & 10 \end{pmatrix} v_2 = \begin{pmatrix} -5 \\ -3 \\ 7 \end{pmatrix}$$

One may take

$$v_2 = \begin{pmatrix} 1/7 \\ 2/7 \\ 0 \end{pmatrix}$$

Next, we solve for v_3 such that

$$(A - \lambda I)v_3 = v_2$$
$$\begin{pmatrix} 1 & -18 & -7 \\ 1 & -11 & -4 \\ -1 & 25 & 10 \end{pmatrix} v_3 = \begin{pmatrix} 1/7 \\ 2/7 \\ 0 \end{pmatrix}$$

One may take

$$v_3 = \begin{pmatrix} 25/49 \\ 1/49 \\ 0 \end{pmatrix}$$

Hence, the general solutions are given by

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = C_1 e^{-2t} \begin{pmatrix} -5 \\ -3 \\ 7 \end{pmatrix} + C_2 e^{-2t} \left(t \begin{pmatrix} -5 \\ -3 \\ 7 \end{pmatrix} + \begin{pmatrix} 1/7 \\ 2/7 \\ 0 \end{pmatrix} \right)$$
$$+ C_3 e^{-2t} \left(\frac{t^2}{2!} \begin{pmatrix} -5 \\ -3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 1/7 \\ 2/7 \\ 0 \end{pmatrix} + \begin{pmatrix} 25/49 \\ 1/49 \\ 0 \end{pmatrix} \right)$$

Remark. *Note that* $\ker(A + 2I)^3 = \mathbb{R}^3$. *Hence, we can take*

$$v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = (A + 2I)v_3 \neq 0, \quad v_1 = (A + 2I)v_2 \neq 0$$

This will simplify the computation. In fact, the choice of v_3 is not completely arbitrary [1], *it has to be in* $\ker(A + 2I)^3 \setminus \ker(A + 2I)^2$. *However, in practice, we may simply choose it arbitrarily. If, for example, we have $v_2 = 0$ after we have chosen v_3, then v_3 will be an eigenvector. Thus, we by accident find an eigenvector. This observation applies to general situation when an $n \times n$ matrix has exactly one eigenvalue and the corresponding eigenspace has dimension one.*

[1] I made a mistake here in the tutorial.