MATH 3270A Tutorial 7

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Theorem 1 (Sturm's Separation Theorem)**.** *Let y_1, y_2 be two linearly independent solutions of the ODE*

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{1}$$

with $p(x)$ and $q(x)$ continuous on (a, b). Then, between any consecutive zeros x_1, x_2 of y_1, there exists exactly one zero of y_2.

Proof. Please refer to the solution of the mid-term. □

Theorem 2. *If $p(x), q(x) \leq 0$, then a non-trivial solution of (1) has at most one zero.*

Proof. We can always rewrite (1) into the following form:

$$(\alpha(x)y'(x))' + \beta(x)y = 0 \tag{2}$$

with $\alpha(x) = e^{\int_0^x p(s)ds} > 0$ and $\beta(x) = \alpha(x)q(x)$. Let y be a non-trivial solution of (2) and suppose there were two zeros $x_1 < x_2$ of y. Without loss of generality, we may suppose $y > 0$ on (x_1, x_2). Then, by integrating (2), we have

$$\begin{aligned} \alpha(x)y'(x)\Big|_{x_1}^{x_2} &= -\int_{x_1}^{x_2} \beta(x)y(x)dx > 0 \\ \implies \quad \alpha(x_2)y'(x_2) &> \alpha(x_1)y'(x_1) \\ \implies \quad y'(x_2) &> y'(x_1) \end{aligned}$$

where the fact $0 < \alpha(x_2) \leq \alpha(x_1)$ has been used in the last step. We have arrived at a contradiction, since $y'(x_2) < 0 < y'(x_1)$ as x_1, x_2 are consecutive zeros and $y > 0$ on (x_1, x_2). □

Theorem 3 (Sturm-Picone Comparison Theorem)**.** *Let $0 < \alpha_1(x) \leq \alpha_2(x)$ and $\beta_2(x) \leq \beta_1(x)$ be continuous functions on (a, b). Let y_1 and y_2 be solutions to the ODEs*

$$\begin{aligned} (\alpha_1(x)y_1'(x))' + \beta_1(x)y_1(x) = 0 \\ (\alpha_2(x)y_2'(x))' + \beta_2(x)y_2(x) = 0 \end{aligned}$$

such that they are linearly independent. Then, between any consecutive zeros x_1, x_2 of y_2, there exists at least one zero of y_1.

Proof. Taking the difference between the product of the first equation and y_2 and the product of the second equation and y_1 gives

$$(\alpha_2 y_1 y_2' - \alpha_1 y_1' y_2)' = \alpha_2 y_1' y_2 - \alpha_1 y_1 y_2' + (\beta_1 - \beta_2)y_1 y_2 \tag{3}$$

Let $x_1 < x_2$ be consecutive zeros of y_2 and suppose there were no zeros of y_1 on (x_1, x_2). Then we have, by (3),

$$\left(\frac{y_2}{y_1}\left(\alpha_2 y_2' y_1 - \alpha_1 y_2 y_1'\right)\right)' = (\beta_1 - \beta_2)y_2^2 + (\alpha_2 - \alpha_1)y_2'^2 + \alpha_2\left(y_2' - y_1'\frac{y_2}{y_1}\right)^2 \geq 0 \tag{4}$$

on (x_1, x_2). Integrating the equality gives

$$0 = \frac{y_2}{y_1}\left(\alpha_2 y_2' y_1 - \alpha_1 y_2 y_1'\right)\Big|_{x_1}^{x_2} = \lim_{x \to x_2}\left(\frac{y_2}{y_1}\left(\alpha_2 y_2' y_1 - \alpha_1 y_2 y_1'\right)\right) - \lim_{x \to x_1}\left(\frac{y_2}{y_1}\left(\alpha_2 y_2' y_1 - \alpha_1 y_2 y_1'\right)\right) \geq 0$$

This implies the RHS of (4) is identically zero, contradicting to the linear independence of y_1 and y_2. $\square$