# MATH 3270A Tutorial 5

Alan Yeung Chin Ching

11th October 2018

## 1 The method of variation of parameters

**Theorem 1** (The method of variation of parameters). *Consider the ODE*

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

*where $p(x)$, $q(x)$ and $f(x)$ are continuous functions on $(a,b)$. Let $\{y_1, y_2\}$ be a set of fundamental solution of the corresponding homogeneous equation of (1). Then, a particular solution is given by*

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

*with*

$$u_1(x) = -\int \frac{y_2(x)f(x)}{W(y_1,y_2)(x)}dx \qquad u_2(x) = \int \frac{y_1(x)f(x)}{W(y_1,y_2)(x)}dx$$

**Exercise.** *Verify the above theorem.*

**Example 1.** *Find the general solutions of the following ODE.*

$$y'' + y = \tan x$$

**Solution**

*Note that $y_1 = \sin x$ and $y_2 = \cos x$ are two fundamental solutions of the homogeneous equation. We compute*

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$$

*Hence,*

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)f(x)}{W(y_1,y_2)(x)}dx = \int (\cos x \tan x)dx = \int \sin x dx = -\cos x + C_1 \\ u_2(x) &= \int \frac{y_1(x)f(x)}{W(y_1,y_2)(x)}dx = -\int (\sin x \tan x)dx = -\int (\sin^2 x \sec x)dx \\ &= \int (\cos^2 x \sec x)dx - \int \sec x dx = \sin x - \ln|\sec x + \tan x| + C \end{aligned}$$

*As a result, one particular solution is given by*

$$y_p(x) = u_1(x)y_x(x) + u_2(x)y_2(x) = -\cos x \log|\sec x + \tan x|$$

*The general solutions are given by $C_1 \cos x + C_2 \sin x - \cos x \log|\sec x + \tan x|$*

**Example 2.** *Find the general solution of the following ODE.*

$$x^2y'' - 2y = 3x^2 - 1, \qquad x > 0$$

**Solution**

*Note that* $y_1 = x^2$ *and* $y_2 = \dfrac{1}{x}$ *are two fundamental solutions of the homogeneous equation. We compute*

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & \frac{1}{x} \\ 2x & -\frac{1}{x^2} \end{vmatrix} = -3$$

*Note that in order to apply Theorem 1, we have to rewrite the ODE as*

$$y'' - \frac{2}{x^2}y = 3 - \frac{1}{x^2}$$

*Hence,*

$$u_1(x) = -\int \frac{y_2(x)f(x)}{W(y_1,y_2)(x)}dx = \int \left(\frac{1}{x} - \frac{1}{3x^3}\right) dx = \log x + \frac{1}{6x^2} + C$$
$$u_2(x) = \int \frac{y_1(x)f(x)}{W(y_1,y_2)(x)}dx = -\int \frac{x^2(3 - \frac{1}{x^2})}{3}dx = -\frac{1}{3}\int (3x^2 - 1)dx = -\frac{x^3}{3} + \frac{x}{3} + C$$

*As a result, one particular solution is given by*

$$y_p(x) = u_1(x)y_x(x) + u_2(x)y_2(x) = x^2 \log x + \frac{1}{2}$$

*The general solutions are given by* $C_1x^2 + C_2\dfrac{1}{x} + x^2 \log x + \dfrac{1}{2}$

**Example 3** (Duhamel's Principle)**.** *Let* $p(x)$*,* $q(x)$ *and* $f(x)$ *be continuous functions on* $(a, b)$ *with* $a < 0 < b$*. Let* $y(x)$ *be the solution of the following IVP.*

$$\begin{cases} y''(x) + p(x)y'(x) + q(x)y(x) &= f(x) \\ y(0) &= \alpha \\ y'(0) &= \beta \end{cases} \tag{2}$$

*Let* $\hat{y}(x)$ *and* $y_s(x)$ *be the solutions of the following IVPs for all* $a < s < b$ *respectively.*

$$\begin{cases} \hat{y}''(x) + p(x)\hat{y}'(x) + q(x)\hat{y}(x) &= 0 \\ \hat{y}(0) &= \alpha \\ \hat{y}'(0) &= \beta \end{cases}$$

$$\begin{cases} y_s''(x) + p(x)y_s'(x) + q(x)y_s(x) &= 0 \\ y_s(s) &= 0 \\ y_s'(s) &= f(s) \end{cases}$$

*Show that*

$$y(x) = \hat{y}(x) + \int_0^x y_s(x)ds$$

*Consequently, give an alternative proof of Theorem 1.*

**Solution**

*Let $Y(x) = \hat{y}(x) + \int_0^x y_s(x)ds$. By uniqueness, it suffices to show that $Y(x)$ is also a solution of (2). We compute that*

$$\begin{aligned}
Y'(x) &= \hat{y}'(x) + \frac{d}{dx}\int_0^x y_s(x)ds \\
&= \hat{y}'(x) + \underbrace{y_x(x)}_{=0} + \int_0^x y_s'(x)ds = \hat{y}'(x) + \int_0^x y_s'(x)ds \\
Y''(x) &= \hat{y}''(x) + \frac{d}{dx}\int_0^x y_s'(x)ds \\
&= \hat{y}''(x) + y_x'(x) + \int_0^x y_s''(x) = \hat{y}''(x) + f(x) + \int_0^x y_s''(x)
\end{aligned}$$

$$\begin{aligned}
& Y''(x) + p(x)Y'(x) + q(x)Y(x) \\
&= \left(\hat{y}''(x) + f(x) + \int_0^x y_s''(x)\right) + p(x)\left(\hat{y}'(x) + \int_0^x y_s'(x)ds\right) + q(x)\left(\hat{y}(x) + \int_0^x y_s(x)ds\right) \\
&= f(x) + (\hat{y}''(x) + p(x)\hat{y}'(x) + q(x)\hat{y}(x)) + \int_0^x (y_s''(x) + p(x)y_s'(x) + q(x)y_s(x))\,ds \\
&= f(x)
\end{aligned}$$

*Clearly, we also have*

$$\begin{aligned}
Y(0) &= \hat{y}(0) + \int_0^0 y_s(0)ds = \alpha \\
Y'(0) &= \hat{y}'(0) + \int_0^0 y_s'(0)ds = \beta
\end{aligned}$$

*Hence, we have $Y(x) = y(x)$ for all $x \in (a,b)$.*
*To prove Theorem 1, let $\{y_1, y_2\}$ be a set of fundamental solutions. We want to find $C_1^s$, $C_2^s$ such that $y_s = C_1^s y_1 + C_2^s y_2$. This can be done by solving the following system.*

$$\begin{cases} C_1^s y_1(s) + C_2^s y_2(s) &= 0 \\ C_1^s y_1'(s) + C_2^s y_2'(s) &= f(s) \end{cases}$$

*Hence, we have*

$$C_1^s = -\frac{f(s)y_2(s)}{W(y_1,y_2)(s)} \qquad C_2^s = \frac{f(s)y_1(s)}{W(y_1,y_2)(s)}$$

*Finally, we note that one particular solution is then given by (taking $\alpha = \beta = 0$)*

$$\begin{aligned}
y(x) &= \int_0^x y_s(x)ds \\
&= \int_0^x (C_1^s y_1(x) + C_2^s y_2(x))ds \\
&= \left(\int_0^x -\frac{f(s)y_2(s)}{W(y_1,y_2)(s)}ds\right) y_1(x) + \left(\int_0^x \frac{f(s)y_1(s)}{W(y_1,y_2)(s)}ds\right) y_2(x) = u_1(x)y_1(x) + u_2(x)y_2(x)
\end{aligned}$$