Suggested solutions to HW4 for MATH3270a

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1. (**3points**) If $x_1 = y$ and $x_2 = y'$, and we consider the second order equation

$$y'' + p(t)y' + q(t)y = 0$$

corresponding to the system of 1^{st} order equations

$$x_1' = x_2,$$
$$x_2' = -q(t)x_1 - p(t)x_2.$$

   (a) (**2points**) Let X be a fundamental matrix for above system and y_1, y_2 be a fundamental set of solutions for above second order equation, **show** that we must have $W[y_1, y_2](t) = c\det(X(t))$ for some non-zero constant c.

   (b) (**1point**) If p, q are constants, by writing the above system as $\frac{dx}{dt} = Ax$ for some 2×2 constant matrix, **show** that the characteristic polyminial of A agrees with the characteristic polynomial of the second order equation.

Solution:

   (a) If y_1, y_2 are a fundamental set of solutions for above second order equation, then $\begin{pmatrix} y_1 \\ y_1' \end{pmatrix}, \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$ are a fundamental set of solutions for above system. Thus there exists some constant matrix C such that

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = X(t)C$$

   then we have

$$0 \neq W[y_1, y_2](t) = \det(X(t))\det(C).$$

   Let $c = \det C$, which cannot be zero since $\det X \neq 0$, so we finish the proof.

   (b) Here $A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$, so the characteristic polynomial is given by $f(r) = r(r+p) + q = r^2 + pr + q$.

2. (**4points**) We consider the system

$$\frac{dy}{dt} = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} y;$$

   (a) (**1point**) **Solve** the above system for $\alpha = \frac{1}{2}$ and classify the critical 0 of the system as to type and stability;

   (b) (**1point**) **Repeat** part (a) for $\alpha = 2$;

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(c) **Find** (**1point**) the eigenvalues of the matrix $\begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix}$ in terms of α, and **determine** (**1point**) the value of α between $\frac{1}{2}$ and 2 where the transition from one behaviour to other occurs.

Solution:

(a) The eigenvalues of $\begin{pmatrix} -1 & -1 \\ -\frac{1}{2} & -1 \end{pmatrix}$ are $\lambda = -1 \pm \frac{\sqrt{2}}{2} < 0$, then critical point 0 is a node and asymptotically stable.

(b) The eigenvalues of $\begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$ are $\lambda_1 = -1 - \sqrt{2} < 0, \lambda_2 = -1 + \sqrt{2} > 0$, then critical point 0 is a saddle point and unstable.

(c) The eigenvalues of $\begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix}$ are $\lambda_1 = -1 - \sqrt{\alpha} < 0, \lambda_2 = -1 + \sqrt{\alpha}$. Hence $\alpha = 1$ is the critical point.

3. (**8points=2points** $\times$ **4**) **Find** the real-valued general solution to the following system of linear differential equations:

(a) $\frac{dy}{dt} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} y + \begin{pmatrix} e^t \\ t \end{pmatrix}$;

(b) $\frac{dy}{dt} = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} y$;

(c) $\frac{dy}{dt} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} y$;

(d) $\frac{dy}{dt} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} y$;

Solution:

(a) The eigenvalue and corresponding eigenvector of matrix $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$ are

$$\lambda_1 = -1, \quad r_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\lambda_2 = 1, \quad r_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix},$$

then the general solution to homogeneous equation is

$$y_c = C_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

with arbitrary constants C_1, C_2. One particular solution is of the form

$$Y(t) = \vec{a}t + \vec{b} + \vec{c}e^t + \vec{d}te^t$$

with constant vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ to be determined. Then

$$\frac{dY}{dt} = \vec{a} + (\vec{c} + \vec{d})e^t + \vec{d}te^t = A\vec{a}t + A\vec{b} + A\vec{c}e^t + A\vec{d}te^t + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

which implies that

$$\begin{cases} A\vec{a} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \\ A\vec{b} = \vec{a}, \\ A\vec{c} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{c} + \vec{d}, \\ A\vec{d} = \vec{d}. \end{cases}$$

Then we have

$$\vec{a} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix}, \quad \vec{d} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

Hence the general solution is given by

$$y = C_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} -3t \\ 2t-1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix} e^t + \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} te^t.$$

(b) The eigenvalue and corresponding eigenvector of matrix $A = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}$ are

$$\lambda_1 = -2, \quad r_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix},$$

$$\lambda_2 = -1 - \sqrt{2}i, \quad r_2 = \begin{pmatrix} \sqrt{2}i \\ -1 \\ \sqrt{2}i + 1 \end{pmatrix},$$

$$\lambda_3 = -1 + \sqrt{2}i, \quad r_2 = \begin{pmatrix} \sqrt{2}i \\ 1 \\ \sqrt{2}i - 1 \end{pmatrix},$$

then the real-valued general solution is

$$y_c = C_1 e^{-2t} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) - \sin(\sqrt{2}t) \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} \sqrt{2}\sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) \\ \sqrt{2}\sin(\sqrt{2}t) + \cos(\sqrt{2}t) \end{pmatrix}$$

with arbitrary constants C_1, C_2, C_3.

(c) The eigenvalue and corresponding eigenvector of matrix $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ are

$$\lambda_1 = -2, \quad r_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix},$$

$$\lambda_2 = 1, \quad r_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix},$$

$$\lambda_3 = 3, \quad r_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

then the general solution is

$$y_c = C_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + C_3 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

with arbitrary constants C_1, C_2, C_3.

(d) The eigenvalue of matrix $A = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}$ is $\lambda = 1$ with algebraic multiplicity 3 and geometric multiplicity 2, the corresponing eigenvectors and generalized eigenvector are

$$r_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$

then the general solution is

$$y_c = C_1 e^{t} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + C_3 e^{t} \left\{ \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

with arbitrary constants C_1, C_2, C_3.

4. (**3points=1point** $\times$ **3**) Let A be a constant $n \times n$ matrix, and we consider the matrix valued differential equation $\Phi' = A\Phi$, with initial value $\Phi(t_0) = B$ for some invertible matrix B.

   (a) **Show** that above initial value problem has a unique soluiton defining on $\mathbb{R}$.

   (b) Suppose $\Phi(t)$ is the unique solution with initial data $\Phi(0) = I$, **show** that $\Phi(t)\Phi(s) = \Phi(t+s)$.

   (c) **Show** that $\Phi(t)\Phi(-t) = I$ and hence $\Phi(t-s) = \Phi(t)\Phi(s)^{-1}$.

Solution:

(a) Let $\Phi(t) = (\phi_1, \phi_2, \cdots, \phi_n)$ with $\phi_i, i, 1, 2, \cdots, n$ vectors, then for any $i \in \{1, 2, \cdots, n\}$

$$\phi_i'(t) = A\phi_i(t)$$
$$\phi_i(0) = b_i$$

where $B = (b_1, b_2, \cdots, b_n)$. Since A is a constant matrix, so the elements of A are continuous on the whole $\mathbb{R}$, then there exists a unique solution ϕ_i for each i by existence and uniqueness of first order system of partial differential equation. Hence there exists a unique solution $\Phi(t) = (\phi_1, \phi_2, \cdots, \phi_n)$ for above initial value problem on whole line $\mathbb{R}$.

(b) Let s be a fixed point. Since $\Phi(t)\Phi(s)$ and $\Phi(t+s)$ satisfy

$$\Phi'(t) = A\Phi(t)$$
$$\Phi(0) = \Phi(s)$$

then by uniqueness they are equal.

(c) Let $s = -t$ in $\Phi(t)\Phi(s) = \Phi(t+s)$, together with $\Phi(0) = I$, we get $\Phi(t)\Phi(-t) = I$ immediately, which implies that $\Phi(-t) = \Phi(t)^{-1}$, thus $\Phi(t-s) = \Phi(t)\Phi(-s) = \Phi(t)\Phi(s)^{-1}$.

5. (**6points=2 points** $\times$ **3**) **Sketch** the phase portrait for each of the linear system of 1^{st} order differential equations:

   (a) $\frac{dy}{dt} = \begin{pmatrix} 1 & 1 \\ -5 & -3 \end{pmatrix} y$;

(b) $\frac{dy}{dt} = \begin{pmatrix} -1 & 0 \\ -1 & -\frac{1}{4} \end{pmatrix} y$;

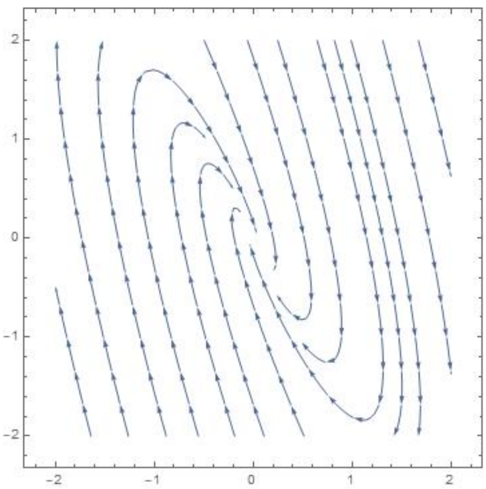
(c) $\frac{dy}{dt} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} y$;

Solution:

(a) First, the critical point of the system is $(0, 0)$.

Then the eigenvalues of matrix $A = \begin{pmatrix} 1 & 1 \\ -5 & -3 \end{pmatrix}$ are $\lambda = -1 \pm i$. So the $(0, 0)$ is stable and a spirial point.

The phase portrait is shown in the following. (Note that the direction at point $(1, 1)$ is $\begin{pmatrix} 2 \\ -8 \end{pmatrix}$, so the trajectory moves in clockwise direction to zero.)



(b) First, the critical point of the system is $(0, 0)$.

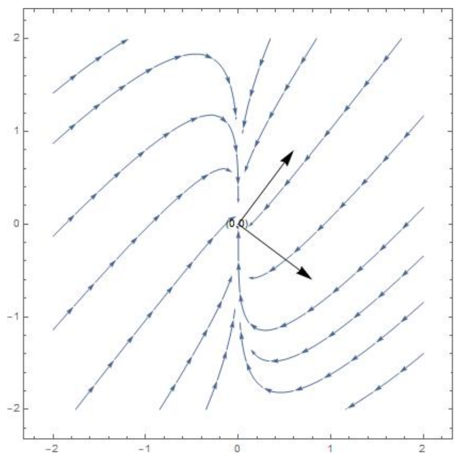
Then the eigenvalue and corresponding eigenvector of matrix $A = \begin{pmatrix} -1 & 0 \\ -1 & -\frac{1}{4} \end{pmatrix}$ are

$$\lambda_1 = -1, \quad r_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\lambda_2 = -\frac{1}{4}, \quad r_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so the point $(0, 0)$ is a node and stable.

The phase portrait is shown in the following (*it should be noted that the eigenvector r_2 is wrong in the picture, please correct it by yourself.*).



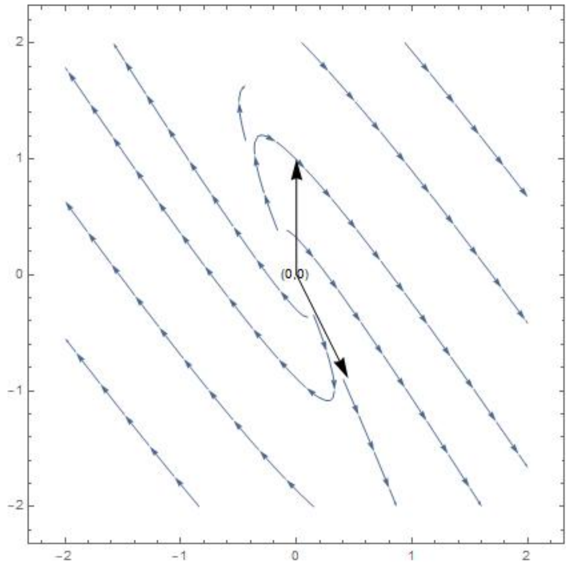
(c) First, the critical point of the system is $(0,0)$.

Then the eigenvalue of matrix $A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$ is $\lambda = 1$ with algebraic multiplicity 2 and geometric multiplicity 1, the corrsponding eigenvector and generalized eigenvector are

$$r = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so the point $(0,0)$ is a node and unstable.

The phase portrait is shown in the following.



6. (**6points=3 points × 2**) For each of the following nonlinear system of 1^{st} order differential equations:

- (**1point**) **Find** all critical points of the above system and the corresponding linear system near each critical points;
- (**1point**) **Determine** the type and stability of the linear system associated to each critical points;
- (**1point**) **Draw** the phase portrait for the nonlinear system of differential equations.

(a)

$$\frac{dy_1}{dt} = (3 + y_1)(y_2 - y_1),$$
$$\frac{dy_2}{dt} = (4 - y_1)(y_2 + y_1);$$

(b)

$$\frac{dy_1}{dt} = y_1(1 - y_1 - y_2),$$
$$\frac{dy_2}{dt} = y_2(2 - y_1 - y_2).$$

Solution:

(a) • Let $F = (3 + y_1)(y_2 - y_1), G = (4 - y_1)(y_2 + y_1)$, then by solving

$$F = 0,$$
$$G = 0,$$

we find all critical points $P_1 = (0,0), P_2 = (-3,3), P_3 = (4,4)$. Then

$$\frac{d(F,G)}{d(y_1,y_2)} = \begin{pmatrix} y_2 - 2y_1 - 3 & 3 + y_1 \\ 4 - y_2 - 2y_1 & 4 - y_1 \end{pmatrix}$$

so the correponding linear system near each critical point is given by

$$y' = A(P_i)y$$

where

$$A(P_1) = \begin{pmatrix} -3 & 3 \\ 4 & 4 \end{pmatrix}, \quad A(P_2) = \begin{pmatrix} 6 & 0 \\ 7 & 7 \end{pmatrix}, \quad A(P_3) = \begin{pmatrix} -7 & 7 \\ -8 & 0 \end{pmatrix}.$$

- The eigenvalues of $A(P_1)$ are $\lambda_1 = \frac{1-\sqrt{97}}{2} < 0, \lambda_1 = \frac{1+\sqrt{97}}{2} > 0$, so the critical point $P_1 = (0,0)$ is a saddle point and unstable. Moreover, the corresponding eigenvectors are

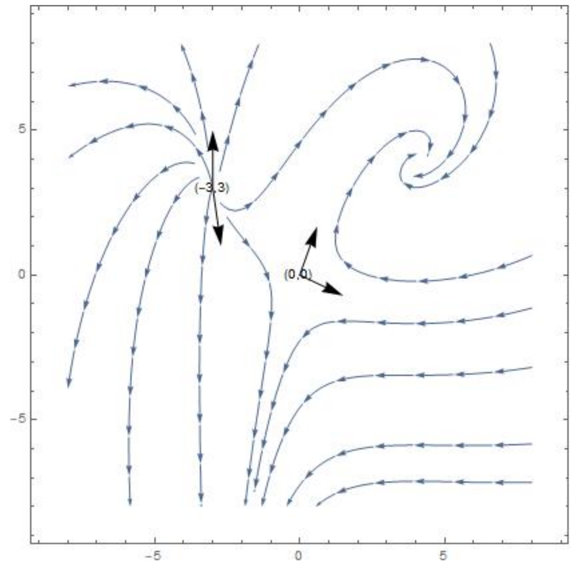
$$r_1 = \begin{pmatrix} 6 \\ 7 - \sqrt{97} \end{pmatrix}, \quad r_2 = \begin{pmatrix} 6 \\ 7 + \sqrt{97} \end{pmatrix}.$$

The eigenvalues of $A(P_2)$ are $\lambda_1 = 6, \lambda_2 = 7$, so the critical point $P_2 = (-3,3)$ is a node and unstable. Moreover, the corresponding eigenvectors are

$$r_1 = \begin{pmatrix} 1 \\ -7 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues of $A(P_3)$ are $\lambda = \frac{-7\pm5\sqrt{7}i}{2}$, so the critical point $P_3 = (4,4)$ is a spirial point and stable. Note that the direction at point $(5,4)$ is $\begin{pmatrix} -7 \\ -40 \end{pmatrix}$, so the trajectory around $P_3 = (4,4)$ moves in clockwise direction to P_3.

- The phase portrait:



(b) - Let $F = y_1(1 - y_1 - y_2), G = y_2(2 - y_1 - y_2)$, then by solving

$$F = 0,$$
$$G = 0,$$

we find all critical points $P_1 = (0,0), P_2 = (0,2), P_3 = (1,0)$. Then

$$\frac{d(F,G)}{d(y_1,y_2)} = \begin{pmatrix} 1 - 2y_1 - y_2 & -y_1 \\ -y_2 & 2 - y_1 - 2y_2 \end{pmatrix}$$

so the correponding linear system near each critical point is given by

$$y' = A(P_i)y$$

where

$$A(P_1) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad A(P_2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}, \quad A(P_3) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

- The eigenvalues of $A(P_1)$ are $\lambda_1 = 1, \lambda_2 = 2$, so the critical point $P_1 = (0,0)$ is a node and unstable. Moreover, the corresponding eigenvectors are

$$r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

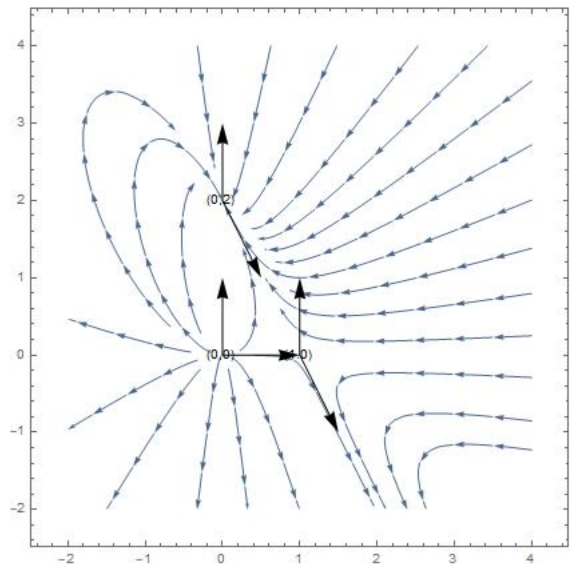
The eigenvalues of $A(P_2)$ are $\lambda_1 = -1, \lambda_2 = -2$, so the critical point $P_2 = (0,2)$ is a node and stable. Moreover, the corresponding eigenvectors are

$$r_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The eigenvalues of $A(P_3)$ are $\lambda_1 = -1, \lambda_2 = 1$, so the critical point $P_3 = (1,0)$ is a saddle point and unstable. Moreover, the corresponding eigenvectors are

$$r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

- The phase portrait:



7. (**4points=2 points × 2**) Use Liapunov's function to show the stability of the following system of differential equations:

   (a) For the system

$$\frac{dy_1}{dt} = -\frac{1}{2}y_1^3 + 2y_1y_2^2,$$
$$\frac{dy_2}{dt} = -2y_2^3,$$

   show that 0 is an asymptotically stable critical point.

(b) For the system

$$\frac{dy_1}{dt} = 2y_1^3 - y_2^3,$$
$$\frac{dy_2}{dt} = 2y_1y_2^2 + 4y_1^2y_2 + 2y_2^3,$$

show that 0 is an unstable critical point.

Solution:

(a) Let

$$V = ay_1^2 + by_1y_2 + cy_2^2,$$

then

$$\begin{aligned}\dot{V} &= (2ay_1 + by_2)\dot{y}_1 + (by_1 + 2cy_2)\dot{y}_2 \\ &= (2ay_1 + by_2)(-\frac{1}{2}y_1^3 + 2y_1y_2^2) + (by_1 + 2cy_2)(-2y_2^3) \\ &= -ay_1^4 - \frac{1}{2}by_1^3y_2 + 4ay_1^2y_2^2 - 4cy_2^4.\end{aligned}$$

If $a = c = 1, b = 0$, then $V = y_1^2 + y_2^2$ is positive definite and $\dot{V} = -y_1^4 + 4y_1^2y_2^2 - 4y_2^4 = -(y_1^2 - 2y_2^2)^2$ is negative semidefinite, thus 0 is a stable critical point.
Moreover, we can show that it's indeed asymptotically stable. For the closed curve (circle for this case) $V = y_1^2 + y_2^2 = c > 0$, since

$$\dot{V} = -(y_1^2 - 2y_2^2)^2 = \begin{cases} > 0, y_1^2 \neq 2y_2^2 \\ = 0, y_1^2 = 2y_2^2 \end{cases}$$

then the direction of the trajectory across this circle at $y_1^2 \neq 2y_2^2$ is inward and if the trajectory aross the circle at these discrete points where $y_1^2 = 2y_2^2$ (at most four points), the direction is tagent to the circle. Hence, there exists a $\delta_0 > 0$ small enough, such that if initial data is in the ball $B_{\delta_0}(0)$, then the trajectory must tend to the origin as time goes to infinity.

(b) Let

$$V = ay_1^2 + by_1y_2 + cy_2^2,$$

then

$$\begin{aligned}\dot{V} &= (2ay_1 + by_2)\dot{y}_1 + (by_1 + 2cy_2)\dot{y}_2 \\ &= (2ay_1 + by_2)(2y_1^3 - y_2^3) + (by_1 + 2cy_2)(2y_1y_2^2 + 4y_1^2y_2 + 2y_2^3) \\ &= 4ay_1^4 + 6by_1^3y_2 + 2(b + 4c)y_1^2y_2^2 + 2(b - a + 2c)y_1y_2^3 + (4c - b)y_2^4.\end{aligned}$$

If $a = 2c = 2, b = 0$, then $V = 2y_1^2 + y_2^2$ is positive definite and $\dot{V} = 8y_1^4 + 8y_1^2y_2^2 + 4y_2^4 = 4(y_1^2 + y_2^2)^2 + 4y_1^4$ is also positive definite, thus 0 is a unstable critical point.