Suggested solutions to HW3 for MATH3270a

Rong ZHANG*

November 9, 2018

1. (**4points=1point $\times$ 4**) **Find** the general solution to the following differential equations:

   (a) $y^{(4)} + 2y^{(3)} + y'' = 0$;

   (b) $y^{(4)} + 8y^{(2)} + 16y = 0$;

   (c) $y^{(3)} + y'' + y' + y = 2e^{-t} + 4t$;

   (d) $y^{(4)} + 2y'' + y = 4 + \cos 2t$;

   Solution:

   (a) The corresponding characteristic equation is

   $$r^4 + 2r^3 + r^2 = 0,$$

   then $r_1 = r_2 = 0, r_3 = r_4 = -1$, so the general solution is

   $$y = C_1 + C_2 t + C_3 e^{-t} + C_4 t e^{-t}$$

   where C_1, C_2, C_3 and C_4 are arbitrary constants.

   (b) The corresponding characteristic equation is

   $$r^4 + 8r^2 + 16 = 0,$$

   then $r = \pm 2i$ with multiplicity 2, so the general solution is

   $$y = C_1 \cos(2t) + C_2 \sin(2t) + C_3 t \cos(2t) + C_4 t \sin(2t)$$

   where C_1, C_2, C_3 and C_4 are arbitrary constants.

   (c) The corresponding characteristic equation is

   $$r^3 + r^2 + r + 1 = 0,$$

   then $r_1 = -1, r_2 = i, r_3 = -i$, so the general solution to the corresponding homogeneous equation is

   $$y_c = C_1 e^{-t} + C_2 \cos t + C_3 \sin t$$

   where C_1, C_2 and C_3 are arbitrary constants. By observation, it's promising to find a particular solution of the form [1]

   $$Y(t) = Ate^{-t} + Bt + C$$

*Any questions on solutions of HW3, please email me at rzhang@math.cuhk.edu.hk

[1]Here we use the method of undetermined coefficients to find a particular solution, you can also try the method of variation of parameters to solve problem 1(c)(d) by yourself.

where constants A, B, C are to be determined. Since

$$\begin{aligned} Y'(t) &= A(1-t)e^{-t} + B, \\ Y''(t) &= A(t-2)e^{-t}, \\ Y^{(3)} &= A(3-t)e^{-t}, \end{aligned}$$

then by substituting $Y(t)$ into the problem, we have

$$2e^{-t} + 4t = Y^{(3)} + Y'' + Y' + Y = 2Ae^{-t} + Bt + C + B$$

which implies that $A = 1, B = 4, C = -4$. Therefore, the general solution is given by

$$y = y_c + Y = C_1e^{-t} + C_2\cos t + C_3\sin t + te^{-t} + 4t - 4$$

with arbitrary constants C_1, C_2 and C_3.

(d) The corresponding characteristic equation is

$$r^4 + 2r^2 + 1 = 0,$$

then $r = \pm i$ with multiplicity 2, so the general solution to the corresponding homogeneous equation is

$$y_c = C_1\cos t + C_2\sin t + C_3t\cos t + C_4t\sin t$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. By observation, it's promising to find a particular solution of the form

$$Y(t) = A + B\cos 2t + C\sin 2t$$

where A, B, C are constants to be determined. Since

$$\begin{aligned} Y'(t) &= -2B\sin 2t + 2C\cos 2t, \\ Y''(t) &= -4B\cos 2t - 4C\sin 2t, \\ Y^{(3)} &= 8B\sin 2t - 8C\cos 2t, \\ Y^{(4)} &= 16B\cos 2t + 16C\sin 2t, \end{aligned}$$

then by substituting $Y(t)$ into the problem, we have

$$4 + \cos 2t = Y^{(4)} + 2Y'' + Y = 9B\cos 2t + 9C\sin 2t + A$$

which implies that $A = 4, B = \frac{1}{9}, C = 0$. Therefore, the general solution is given by

$$y = y_c + Y = C_1\cos t + C_2\sin t + C_3t\cos t + C_4t\sin t + 4 + \frac{1}{9}\cos 2t$$

with arbitrary constants C_1, C_2 and C_3.

2. (**1points=0.5points $\times$ 2**) **Determine** a suitable form of $Y(t)$ for using the method of undetermined coefficients to the following equations:

   (a) $y^{(3)} - 2y'' + y' = 3t^3 + 2e^t$;

   (b) $y^{(4)} - y^{(3)} - y'' + y' = t^2 + 8 + t\sin t$.

Solution:

(a) The corresponding characteristic equation is

$$r^3 - 2r^2 + r = 0,$$

then $r_1 = 0, r_2 = r_3 = 1$, so one particular solution is of the form

$$Y(t) = (At^3 + Bt^2 + Ct + D)t + Et^2e^t$$

where A, B, C, D and E are constants to be determined.

(b) The corresponding characteristic equation is

$$r^4 - r^3 - r^2 + r = 0,$$

then $r_1 = 0, r_2 = r_3 = 1, r_4 = -1$, so one particular solution is of the form

$$Y(t) = (At^2 + Bt + C)t + (Dt + E)\cos t + (Ft + G)\sin t$$

where A, B, C, D, E, F and G are constants to be determined.

3. (**2point**) **Write** down a formula involving integrals for a particular solution $Y(t)$ of the differential equation

$$y^{(3)} - 3y'' + 3y' - y = r(t),$$

and use it to **solve** for $Y(t)$ when $r(t) = t^{-2}e^t$.

Solution: The corresponding characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = 0,$$

so $r_1 = r_2 = r_3 = 1$ and a fundamental set of solution to homogeneous equation is

$$\{y_1 = e^t, y_2 = te^t, y_3 = t^2e^t\},$$

and

$$W[y_1, y_2, y_3](t) = \begin{vmatrix} e^t & te^t & t^2e^t \\ e^t & (t+1)e^t & (t^2+2t)e^t \\ e^t & (t+2)e^t & (t^2+4t+2)e^t \end{vmatrix} = 2e^{3t}.$$

Then we intend to find a particular solution to the non-homogeneous equation of the following form

$$Y(t) = C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t,$$

with functions $C_1(t), C_2(t)$ and $C_3(t)$ to be determined. Since

$$\begin{aligned} Y'(t) =& \Big(C_1'(t)y_1 + C_2'(t)y_2 + C_3'(t)y_3\Big) + C_1(t)y_1' + C_2(t)y_2' + C_3(t)y_3', \\ Y''(t) =& \frac{d}{dt}\Big(C_1'(t)y_1 + C_2'(t)y_2 + C_3'(t)y_3\Big) + C_1'(t)y_1' + C_2'(t)y_2' + C_3'(t)y_3' \\ &+ C_1(t)y_1'' + C_2(t)y_2'' + C_3(t)y_3'', \\ Y^{(3)} =& \frac{d^2}{dt^2}\Big(C_1'(t)y_1 + C_2'(t)y_2 + C_3'(t)y_3\Big) + \frac{d}{dt}\Big(C_1'(t)y_1' + C_2'(t)y_2' + C_3'(t)y_3'\Big) \\ &+ C_1'(t)y_1'' + C_2'(t)y_2'' + C_3'(t)y_3'' \\ &+ C_1(t)y_1''' + C_2(t)y_2''' + C_3(t)y_3''', \end{aligned}$$

so if $C_1'(t)$ and $C_2'(t)$ satisfy the following algebriac system

$$\begin{aligned} C_1'(t)y_1 + C_2'(t)y_2 + C_3'(t)y_3 &= 0, \\ C_1'(t)y_1' + C_2'(t)y_2' + C_3'(t)y_3' &= 0, \\ C_1'(t)y_1'' + C_2'(t)y_2'' + C_3'(t)y_3'' &= r(t), \end{aligned}$$

then $Y(t)$ is a solution to $y^{(3)} - 3y'' + 3y' - y = r(t)$. By solving the above system, we have

$$C_1'(t) = r(t)\frac{W_1}{W[y_1, y_2, y_3]},$$
$$C_2'(t) = r(t)\frac{W_2,}{W[y_1, y_2, y_3]},$$
$$C_3'(t) = r(t)\frac{W_3}{W[y_1, y_2, y_3]},$$

where

$$W_1 = \begin{vmatrix} 0 & te^t & t^2e^t \\ 0 & (t+1)e^t & (t^2+2t)e^t \\ 1 & (t+2)e^t & (t^2+4t+2)e^t \end{vmatrix} = t^2e^{2t},$$
$$W_2 = \begin{vmatrix} e^t & 0 & t^2e^t \\ e^t & 0 & (t^2+2t)e^t \\ e^t & 1 & (t^2+4t+2)e^t \end{vmatrix} = -2te^{2t},$$
$$W_3 = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix} = e^{2t}.$$

Thus a particular solution is of the following form

$$\begin{aligned} Y(t) &= C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t \\ &= e^t \int \frac{1}{2}r(t)t^2e^{-t}dt + te^t \int -r(t)te^{-t}dt + t^2e^t \int \frac{1}{2}r(t)e^{-t}dt. \end{aligned}$$

If $r(t) = t^{-2}e^t$, then

$$\begin{aligned} Y(t) &= C_1(t)e^t + C_2(t)te^t + C_3(t)t^2e^t \\ &= e^t \int \frac{1}{2}dt + te^t \int -t^{-1}dt + t^2e^t \int \frac{1}{2}t^{-2}dt \\ &= -t\ln te^t. \end{aligned}$$

4. (**3points**) If y_1 is a solution to the equation

$$y^{(3)} + p_2(t)y'' + p_1(t)y' + p_0(t)y = 0,$$

(a) (**1point**) use the substitution $y = y_1(t)v(t)$ to **derive** a second order ODE for $u = v'$;

(b) (**2points**) use it to **find** the general solution for

$$(2-t)y^{(3)} + (2t-3)y'' - ty' + y = 0,$$

for $t < 2$ provided that $y_1(t) = e^t$ is a solution.

Solution:

(a) Let

$$y = y_1(t)v(t),$$

then $y' = y_1'v + y_1v', y'' = y_1''v + 2y_1'v' + y_1v''$ and $y^{(3)} = y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v'''$. So the original ODE becomes

$$y_1v''' + (3y_1' + p_2y_1)v'' + (3y_1'' + 2p_2y_1' + p_1y_1)v' + (y_1''' + p_2y_1'' + p_1y_1' + p_0y_1)v = 0.$$

Since y_1 is a solution, then $u = v'$ satisfies

$$y_1u'' + (3y_1' + p_2y_1)u' + (3y_1'' + 2p_2y_1' + p_1y_1)u = 0,$$

which is a second order ODE for u.

(b) In this case, $y_1 = e^t$ and

$$p_2 = \frac{2t-3}{2-t},\ p_1 = \frac{-t}{2-t},\ p_0 = \frac{1}{2-t}.$$

Let $y = y_1 v$, we have

$$v''' + \frac{3-t}{2-t}v'' = 0,$$

which is a first order linear ODE of v'', then

$$v'' = C_1 e^{-\int \frac{3-t}{2-t}dt} = C_1(t-2)e^{-t}$$

and then

$$v' = C_1 \int (t-2)e^{-t}dt + C_2 = -C_1(t-1)e^{-t} + C_2$$

and thus

$$v = -C_1 \int (t-1)e^{-t}dt + C_2 t + C_3 = C_1 te^{-t} + C_2 t + C_3$$

where C_1, C_2, C_3 are arbitrary constants. Finally, the general solution is given by

$$y = C_1 t + C_2 te^t + C_3 e^t.$$

5. (**5points=1point** $\times$ **5**)[Method of Annihilators] Consider the following differential equation

$$(\frac{d}{dt} - 2)^3(\frac{d}{dt} + 1)y = 3e^{2t} - te^{-t} \tag{1}$$

and try to solve a particular solution $Y(t)$.

(a) **Show** that the linear differential operators with constant coefficients obey the commutative law

$$(\frac{d}{dt} - a)(\frac{d}{dt} - b)f = (\frac{d}{dt} - b)(\frac{d}{dt} - a)f$$

for any twice-differentiable f and any constants a, b.

(b) **Show** that the operator $(\frac{d}{dt} - 2)(\frac{d}{dt} + 1)^2$ annihilates the terms on the R.H.S. of equation (1).

(c) By applying $(\frac{d}{dt} - 2)(\frac{d}{dt} + 1)^2$ to both side of equation (1), **show** that a particular solution $Y(t)$ should satisfy

$$(\frac{d}{dt} - 2)^4(\frac{d}{dt} + 1)^3 Y = 0$$

and hence **show** that

$$Y = c_1 e^{2t} + c_2 te^{2t} + c_3 t^2 e^{2t} + c_4 t^3 e^{2t} + c_5 e^{-t} + c_6 te^{-t} + c_7 t^2 e^{-t}$$

for some constants $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ to be determined.

(d) **Solve** a particular solution $Y(t)$ for equation (1).

(e) For each of $P_k(t), e^{\alpha t}P_k(t), e^{\alpha t}\sin(\mu t)P_k(t), e^{\alpha t}\cos(\mu t)P_k(t)$, **write down** a polynomial $f(r)$ such that $f(\frac{d}{dt})$ is an annihilator for it.

Solution:

(a) It follows from simple calculations that

$$(\frac{d}{dt}-a)(\frac{d}{dt}-b)f=\frac{d^2}{dt^2}f-(a+b)\frac{d}{dt}f+ab=(\frac{d}{dt}-b)(\frac{d}{dt}-a)f.$$

(b) Note that

$$(\frac{d}{dt}-2)(\frac{d}{dt}+1)^2(3e^{2t}-te^{-t})=0.$$

(c) It's obvious.

(d) Since $e^{2t}, te^{2t}, t^2e^{2t}$ and e^{-t} are solutions to the corresponding homogeneous equation, then $c_1=c_2=c_3=c_5=0$, and then

$$\begin{aligned}
3e^{2t}-te^{-t} &= (\frac{d}{dt}-2)^3(\frac{d}{dt}+1)Y=(\frac{d}{dt}-2)^3(\frac{d}{dt}+1)(c_4t^3e^{2t}+c_6te^{-t}+c_7t^2e^{-t})\\
&=(\frac{d}{dt}-2)^3(3c_4t^2e^{2t}+3c_4t^3e^{2t}+c_6e^{-t}+2c_7te^{-t})\\
&=(\frac{d}{dt}-2)^3(3c_4t^3e^{2t}+c_6e^{-t}+2c_7te^{-t})\\
&=18c_4e^{2t}+(\frac{d}{dt}-2)^3(c_6e^{-t}+2c_7te^{-t})\\
&=18c_4e^{2t}+(\frac{d}{dt}-2)^2\big((2c_7-3c_6)e^{-t}-6c_7te^{-t}\big)\\
&=18c_4e^{2t}+(\frac{d}{dt}-2)\big((9c_6-12c_7)e^{-t}+18c_7te^{-t}\big)\\
&=18c_4e^{2t}+(54c_7-27c_6)e^{-t}-54c_7te^{-t}
\end{aligned}$$

which implies that $c_4=\frac{1}{6}, c_6=\frac{1}{27}, c_7=\frac{1}{54}$, so

$$Y=\frac{1}{6}t^3e^{2t}+\frac{1}{27}te^{-t}+\frac{1}{54}t^2e^{-t}.$$

(e)

$$\begin{aligned}
P_k(t): &\qquad f(t)=t^{k+1},\\
e^{\alpha t}P_k(t): &\qquad f(t)=(t-\alpha)^{k+1},\\
e^{\alpha t}\sin(\mu t)P_k(t): &\qquad f(t)=(t^2-2\alpha t+\alpha^2+\mu^2)^{k+1},\\
e^{\alpha t}\cos(\mu t)P_k(t): &\qquad f(t)=(t^2-2\alpha t+\alpha^2+\mu^2)^{k+1}.
\end{aligned}$$