Suggested solutions to HW2 for MATH3270a

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October 12, 2018

1. (**4points=0.5points × 8**) Find the general solution to the following differential equations:

   (a) $y'' + 8y' - 9y = 0$;
   (b) $9y'' + 16y = 0$;
   (c) $y'' + 4y' + 4y = 0$;
   (d) $y'' - 2y' + y = 4e^{-t}$;
   (e) $y'' + 2y' - 3y = 3te^{2t}$;
   (f) $2y'' + 3y' + y = t^2 + 3\cos t$;
   (g) $y'' + 2y' + 5y = 4e^{-t}\cos 2t$;
   (h) $t^2y'' + 7ty' + 5y = 3t$, for $t > 0$ provided that $y_1 = t^{-1}$ is a solution to the corresponding homogeneous equation.

   Solution:

   (a) The corresponding characteristic equation is
   $$r^2 + 8r - 9 = 0,$$
   then $r_1 = -9$ and $r_2 = 1$, so the general solution is
   $$y = C_1e^{-9t} + C_2e^t$$
   where C_1 and C_2 are arbitrary constants.

   (b) The corresponding characteristic equation is
   $$9r^2 + 16 = 0,$$
   then $r = \pm\frac{4}{3}i$, so the general solution is
   $$y = C_1\cos(\frac{4}{3}t) + C_2\sin(\frac{4}{3}t)$$
   where C_1 and C_2 are arbitrary constants.

   (c) The corresponding characteristic equation is
   $$r^2 + 4r + 4 = 0,$$
   then $r_1 = r_2 = -2$, so the general solution is
   $$y = C_1e^{-2t} + C_2te^{-2t}$$
   where C_1 and C_2 are arbitrary constants.

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(d) The corresponding characteristic equation is

$$r^2 + 2r + 1 = 0,$$

then $r_1 = r_2 = -1$, so the general solution to the corresponding homogeneous equation is

$$y_c = C_1 e^{-t} + C_2 t e^{-t}$$

where C_1 and C_2 are arbitrary constants. By observation, it's promising to find a particular solution of the form [1]

$$Y(t) = At^2 e^{-t}$$

where constant A is to be determined. Since

$$Y'(t) = A(2t - t^2)e^{-t},$$
$$Y''(t) = A(2 - 4t + t^2)e^{-t},$$

then by substituting $Y(t)$ into the problem, we have

$$4e^{-t} = Y'' + 2Y' + Y = 2Ae^{-t}$$

which implies that $A = 2$. Therefore, the general solution is given by

$$y = y_c + Y = C_1 e^{-t} + C_2 t e^{-t} + 2t^2 e^{-t}$$

with arbitrary constants C_1 and C_2.

(e) The corresponding characteristic equation is

$$r^2 - 2r - 3 = 0,$$

then $r_1 = -1, r_2 = 3$, so the general solution to the corresponding homogeneous equation is

$$y_c = C_1 e^{-t} + C_2 e^{3t}$$

where C_1 and C_2 are arbitrary constants. By observation, it's promising to find a particular solution of the form

$$Y(t) = (A + Bt)e^{2t}$$

where A, B are constants to be determined. Since

$$Y'(t) = (B + 2A + 2Bt)e^{2t},$$
$$Y''(t) = (4B + 4A + 4Bt)e^{2t},$$

then by substituting $Y(t)$ into the problem, we have

$$3te^{2t} = Y'' - 2Y' - 3Y = (2B - 3A - 3Bt)e^{2t}$$

which implies that $A = -\frac{2}{3}, B = -1$. Therefore, the general solution is given by

$$y = y_c + Y = C_1 e^{-t} + C_2 e^{3t} - (\frac{2}{3} + t)e^{2t}$$

with arbitrary constants C_1 and C_2.

[1] Here we use the method of undetermined coefficients to find a particular solution, you can also try the method of variation of parameters to solve problem 1(d)(e)(f)(g) by yourself.

(f) The corresponding characteristic equation is

$$2r^2 + 3r + 1 = 0,$$

then $r_1 = -1, r_2 = -\frac{1}{2}$, so the general solution to the corresponding homogeneous equation is

$$y_c = C_1 e^{-t} + C_2 e^{-\frac{1}{2}t}$$

where C_1 and C_2 are arbitrary constants. By observation, it's promising to find a particular solution of the form

$$Y(t) = At^2 + Bt + C + D\cos t + E\sin t$$

where constants A, B, C, D, E are to be determined. Since

$$\begin{aligned} Y'(t) &= 2At + B - D\sin t + E\cos t, \\ Y''(t) &= 2A - D\cos t - E\sin t, \end{aligned}$$

then by substituting $Y(t)$ into the problem, we have

$$\begin{aligned} t^2 + 3\cos t &= 2Y'' + 3Y' + Y \\ &= At^2 + (6A + B)t + 4A + 3B + C + (3E - D)\cos t - (E + 3D)\sin t \end{aligned}$$

which implies that $A = 1, B = -6, C = 14, D = -\frac{3}{10}, E = \frac{9}{10}$. Therefore, the general solution is given by

$$y = y_c + Y = C_1 e^{-t} + C_2 e^{-\frac{1}{2}t} + t^2 - 6t + 14 - \frac{3}{10}\cos t + \frac{9}{10}\sin t$$

with arbitrary constants C_1 and C_2.

(g) The corresponding characteristic equation is

$$r^2 + 2r + 5 = 0,$$

then $r = -1 \pm 2i$, so the general solution to the corresponding homogeneous equation is

$$y_c = (C_1\cos 2t + C_2\sin 2t)e^{-t}$$

where C_1 and C_2 are arbitrary constants. By observation, it's promising to find a particular solution of the form

$$Y(t) = t(A\cos 2t + B\sin 2t)e^{-t}$$

where constants A, B are to be determined. Since

$$\begin{aligned} Y'(t) =& (A\cos 2t + B\sin 2t)e^{-t} + \big[(2B - A)\cos 2t - (2A + B)\sin 2t\big]te^{-t}, \\ Y''(t) =& 2\big[(2B - A)\cos 2t - (2A + B)\sin 2t\big]e^{-t} \\ &+ \big[-(3A + 4B)\cos 2t + (4A - 3B)\sin 2t\big]te^{-t}, \end{aligned}$$

then by substituting $Y(t)$ into the problem, we have

$$\begin{aligned} 4e^{-t}\cos 2t &= Y'' + 2Y' + 5Y \\ &= 4(B\cos 2t - A\sin 2t)e^{-t}, \end{aligned}$$

which implies that $A = 0, B = 1$. Therefore, the general solution is given by

$$y = y_c + Y = (C_1\cos 2t + C_2\sin 2t)e^{-t} + t\sin 2te^{-t}$$

with arbitrary constants C_1 and C_2.

(h) It's noted that $y_1 = t^{-1}$ is a solution, then consider [2]

$$z = ty,$$

we have $z' = ty' + y$ and $z'' = ty'' + 2y'$. So the original ODE becomes

$$tz'' + 5z' = 3t.$$

Multiplying the above equation by t^4, we we have

$$\frac{d}{dt}(t^5 z') = 3t^5$$

which implies that

$$t^5 z' = \frac{1}{2}t^6 - 4C_1$$

and thus

$$z = \frac{1}{4}t^2 + C_1 t^{-4} + C_2,$$

where C_1, C_2 are arbitrary constants. Finally, the general solution is given by

$$y = \frac{1}{4}t + C_1 t^{-5} + C_2 t^{-1}.$$

2. (**1points=0.5points $\times$ 2**) Determine a suitable form of $Y(t)$ for using the method of undetermined coefficients to the following equations:

   (a) $y'' - 4y' + 4y = 4t^2 + 4te^{2t} + t\sin 2t$;

   (b) $y'' + 3y' + 2y = e^t(t^2 + 1)\sin 2t + 3e^{-t}\cos t + 6e^t$.

   Solution:

   (a) The corresponding characteristic equation is

   $$r^2 - 4r + 4 = 0,$$

   then $r_1 = r_2 = 2$, so one particular solution is of the form

   $$Y(t) = At^2 + Bt + C + (Dt + E)t^2 e^{2t} + (Ft + G)\sin 2t + (Ht + I)\cos 2t$$

   where A, B, C, D, E, F, G, H and I are constants to be determined.

   (b) The corresponding characteristic equation is

   $$r^2 + 3r + 2 = 0,$$

   then $r_1 = -1, r_2 = -2$, so one particular solution is of the form

   $$\begin{aligned} Y(t) = (A + Bt + Ct^2)e^t \sin 2t + (D + Et + Ft^2)e^t \cos 2t \\ +(G\cos t + H\sin t)e^{-t} + Ie^t \end{aligned}$$

   where A, B, C, D, E, F, G, H and I are constants to be determined.

3. (**1point**) If $y(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = r(t)$, where $r(t)$ is not always zero, show that $cy(t)$ is not a solution for the equation for $c \neq 1$.

   Solution: It follows from direct computications that

   $$(cy)'' + p(t)(cy)' + q(t)(cy) = c(y'' + p(t)y' + q(t)y) = cr(t) \neq r(t)$$

   if $r(t)$ is not always zero and $c \neq 1$.

[2] You can also use Abel's theorem to find another solution to the corresponding homogeneous problem and then use variation of parameters method to find a particular solution for inhomogeneous problem.

4. (**1point**) Can $y(t) = \sin t^2$ be a solution on an interval $I = (-\delta, \delta)$ to the equation $y'' + p(t)y' + q(t)y = 0$ with continuous coefficients $p(t)$ and $q(t)$? **Explain** your answer.

   Solution: No; In fact, suppose $y(t) = \sin t^2$ is a solution then it satisfies the conditions $y(0) = 0, y'(0) = 0$. It's obvious that $y \equiv 0$ is a solution to

$$\begin{aligned} y'' + p(t)y' + q(t)y = 0, \\ y(0) = 0, y'(0) = 0. \end{aligned}$$

   Since all the coefficients $p(t), q(t)$ are continuous, then there exists a unique solution on some interval $I = (-\delta, \delta)$ to this problem by Existence and Uniqueness Theorem. However, $\sin t^2 \neq 0$, which shows that $y(t) = \sin t^2$ can not be a solution on some neighborhood of 0.

5. (**1point**) Assume that y_1 and y_2 is a fundamental set of solutions to $y'' + p(t)y' + q(t)y = 0$, and we let $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$, where a_1, a_2, b_1, b_2 are constants. **Show (0.5points)** that

$$W(y_3, y_4) = (a_1b_2 - a_2b_1)W(y_1, y_2),$$

   and **determine (0.5points)** the condition that y_3, y_4 is again a fundamental set of solution.

   Solution: It follows from direct computications that

$$\begin{aligned} W(y_3, y_4) &= \begin{vmatrix} y_3 & y_4 \\ y_3' & y_4' \end{vmatrix} \\ &= \begin{vmatrix} a_1y_1 + a_2y_2 & b_1y_1 + b_2y_2 \\ a_1y_1' + a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} \\ &= \begin{vmatrix} a_1y_1 & b_1y_1 + b_2y_2 \\ a_1y_1' & b_1y_1' + b_2y_2' \end{vmatrix} + \begin{vmatrix} a_2y_2 & b_1y_1 + b_2y_2 \\ a_2y_2' & b_1y_1' + b_2y_2' \end{vmatrix} \\ &= \begin{vmatrix} a_1y_1 & b_2y_2 \\ a_1y_1' & b_2y_2' \end{vmatrix} + \begin{vmatrix} a_2y_2 & b_1y_1 \\ a_2y_2' & b_1y_1' \end{vmatrix} \\ &= a_1b_2 \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} + a_2b_1 \begin{vmatrix} y_2 & y_1 \\ y_2' & y_1' \end{vmatrix} \\ &= (a_1b_2 - a_2b_1)W(y_1, y_2). \end{aligned}$$

   It's noted that any two solutions y_1, y_2 form a fundamental set iff their Wronskian is nonzero $W(y_1, y_2) \neq 0$, so y_3, y_4 is again a fundamental set of solution iff

$$a_1b_2 - a_2b_1 \neq 0.$$

6. (**1point**) Given f, g, h be continuously differentiable function on an interval I, show that

$$W(fg, fh) = f^2W(g, h).$$

   Solution: It follows from direct computications that

$$\begin{aligned} W(fg, fh) &= \begin{vmatrix} fg & fh \\ f'g + fg' & f'h + fh' \end{vmatrix} \\ &= \begin{vmatrix} fg & fh \\ fg' & fh' \end{vmatrix} \\ &= f^2W(g, h). \end{aligned}$$

7. (**2points=1point** $\times$ **2**) Consider an equation of the form

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, \tag{1}$$

   for $t > 0$, where α, β are real constants.

(a) Using the substitution $x = \log t$, **compute** the terms $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$.

(b) Use the above subsitution to **transform** the equation into

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0, \tag{2}$$

and **conclude** that $y_1(\log t), y_2(\log t)$ is a fundamental set of solutions to equation (1) if $y_1(x), y_2(x)$ is a fundamental set of solutions to equation (2).

Solution:

(a) It follows from chain rule and $\frac{dx}{dt} = \frac{1}{t}$ that

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} = t\frac{dy}{dt},\\ \frac{d^2y}{dx^2} &= \frac{d}{dx}(t\frac{dy}{dt}) = t\frac{d}{dt}(t\frac{dy}{dt})\\ &= t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}.\end{aligned}$$

(b) First, substituting the above relationship into (1) yields (2) directly.
Then it's obvious that $y_1(\log t), y_2(\log t)$ are solutions to (1), so it suffices to show that their Wronskian is nonzero. In fact,

$$\begin{aligned}W(y_1(\log t), y_2(\log t)) &= \begin{vmatrix} y_1(\log t) & y_2(\log t) \\ \frac{d}{dt}y_1(\log t) & \frac{d}{dt}y_2(\log t)\end{vmatrix}\\ &= \begin{vmatrix} y_1(\log t) & y_2(\log t) \\ \frac{1}{t}\frac{d}{dx}y_1(\log t) & \frac{1}{t}\frac{d}{dx}y_2(\log t)\end{vmatrix}\\ &= \frac{1}{t}\begin{vmatrix} y_1(\log t) & y_2(\log t) \\ \frac{d}{dx}y_1(\log t) & \frac{d}{dx}y_2(\log t)\end{vmatrix}\\ &= \frac{1}{t}W(y_1(x), y_2(x)) \neq 0.\end{aligned}$$

8. (**2points=1point** $\times$ **2**) Consider the equation

$$ay'' + by' + cy = 0.$$

(a) If all a, b, c are positive constants, **show** that all the solutions of the equation approach 0 as $t \to \infty$.

(b) If $a > 0, c > 0$ but $b = 0$, **show** that all the solutions are bounded as $t \to \infty$.

Solution:

(a) The corresponding characteristic equation

$$ar^2 + br + c = 0,$$

so $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

- If $b^2 - 4ac > 0$, the general solution is given by

$$y = C_1e^{\frac{-b+\sqrt{b^2-4ac}}{2a}t} + C_2e^{-b-\frac{\sqrt{b^2-4ac}}{2a}t}.$$

  Since $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a} < 0$, so $y \to 0$ as $t \to \infty$;

- If $b^2 - 4ac = 0$, the general solution is given by

$$y = (C_1 + C_2t)e^{-\frac{b}{2a}t},$$

  which implies that $y \to 0$ as $t \to \infty$;

- If $b^2 - 4ac < 0$, the general solution is given by

$$y = \big(C_1 \cos(\frac{\sqrt{4ac - b^2}}{2a}t) + C_2 \sin(\frac{\sqrt{4ac - b^2}}{2a}t)\big)e^{-\frac{b}{2a}t}$$

which implies that $y \to 0$ as $t \to \infty$.

(b) If $a > 0, c > 0$ but $b = 0$, the general solution is given by

$$y = C_1 \cos(\frac{\sqrt{4ac}}{2a}t) + C_2 \sin(\frac{\sqrt{4ac}}{2a}t)$$

which implies that for any t,

$$|y(t)| \le |C_1| + |C_2|.$$

9. (**2points=1point** $\times$ **2**) Consider the differential equation

$$y'' + y = r(t).$$

(a) **Deduce** that

$$Y(t) := \int_{t_0}^{t} \sin(t - s)r(s)ds$$

is a solution to the initial value problem $y(t_0) = 0$, and $y'(t_0) = 0$ using integral formula obtained from method of variation of parameter for non-homogeneous equation.

(b) **Find** the solution of the initial value problem $y(t_0) = 1$, and $y'(t_0) = 2$ in terms of the particular solution $Y(t)$.

Solution:

(a) The corresponding characteristic equation is

$$r^2 + 1 = 0,$$

so $r = \pm i$ and the general solution to the corresponding homogeneous equation is

$$y_c = C_1 \cos t + C_2 \sin t,$$

with constants C_1 and C_2. Then we intend to find a particular solution to the non-homogeneous equation of the following form

$$Y(t) = C_1(t) \cos t + C_2(t) \sin t,$$

with functions $C_1(t)$ and $C_2(t)$ to be determined. Since

$$Y'(t) = \big(C_1'(t) \cos t + C_2'(t) \sin t\big) + C_2(t) \cos t - C_1(t) \sin t,$$
$$Y''(t) = \frac{d}{dt}\big(C_1'(t) \cos t + C_2'(t) \sin t\big) + (C_2'(t) - C_1(t)) \cos t - (C_1'(t) + C_2(t)) \sin t,$$

so if $C_1'(t)$ and $C_2'(t)$ satisfy the following algebriac system

$$C_1'(t) \cos t + C_2'(t) \sin t = 0,$$
$$-C_1'(t) \sin t + C_2'(t) \cos t = r(t),$$

then $Y(t)$ is a solution to $y'' + y = r(t)$. By solving the above system, we have

$$C_1'(t) = -r(t) \sin t,$$
$$C_2'(t) = r(t) \cos t.$$

Note that the initial data $y(t_0) = 0, y'(t_0) = 0$ show that

$$\begin{aligned} C_1(t_0)\cos t_0 + C_2(t_0)\sin t_0 = 0, \\ C_2(t_0)\cos t_0 - C_1(t_0)\sin t_0 = 0, \end{aligned}$$

thus

$$C_1(t_0) = C_2(t_0) = 0.$$

Then solving the equations of $C_1(t)$ and $C_2(t)$ gives

$$\begin{aligned} C_1(t) &= -\int_{t_0}^{t} r(s)\sin s ds, \\ C_2(t) &= \int_{t_0}^{t} r(s)\cos s ds. \end{aligned}$$

Finally, we can find a particular solution

$$Y(t) = C_1(t)\cos t + C_2(t)\sin t = \int_{t_0}^{t} \sin(t-s)r(s)ds.$$

(b) First, we find the solution to the following initial value problem

$$\begin{aligned} &y_1'' + y_1 = 0, \\ &y_1(t_0) = 1, y_1'(t_0) = 2, \end{aligned}$$

which is given by

$$y_1 = (\cos t_0 - 2\sin t_0)\cos t + (2\cos t_0 + \sin t_0)\sin t = \cos(t - t_0) + 2\sin(t - t_0).$$

Then, note that $Y(t) = \int_{t_0}^{t} \sin(t-s)r(s)ds$ is the solution to

$$\begin{aligned} &y_2'' + y_2 = r(t), \\ &y_2(t_0) = 0, y_2'(t_0) = 0. \end{aligned}$$

Hence,

$$y(t) = y_1 + Y(t) = \cos(t - t_0) + 2\sin(t - t_0) + \int_{t_0}^{t} \sin(t-s)r(s)ds.$$

is the solution to

$$\begin{aligned} &y'' + y = r(t), \\ &y(t_0) = 1, y'(t_0) = 2. \end{aligned}$$