HOMEWORK IV (DEADLINE : 7TH DECEMBER, 2018)

ORDINARY DIFFERENTIAL EQUATIONS

Answer all questions:

(1) (3 points) If $x_1 = y$ and $x_2 = y'$, and we consider the second order equation

$$y'' + p(t)y' + q(t)y = 0$$

corresponding to the system of 1^{st} order equations

$$x_1' = x_2,$$
$$x_2' = -q(t)x_1 - p(t)x_2.$$

(a) If we have a fundamental matrix $X(t)$ for the above system of equation, and a fundamental set of solutions y_1, y_2 for the above second order equation, **show** that we must have $W(y_1, y_2) = c\det(X(t))$ for some non-zero constant c.

(b) In the case that $p(t) = p$ and $q(t) = q$ being constants, by writing the above system of 1^{st} order equations as $\frac{d\vec{x}}{dt} = A \cdot \vec{x}$ for some 2×2 constant matrix A, **show** that the characteristic polynomial of A agrees with the characteristic polynomial $f(r) = r^2 + pr + q$ of the second order equation.

(2) (4 points) We consider the system

$$\frac{d\vec{y}}{dt} = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \cdot \vec{y}.$$

(a) **Solve** the above system for $\alpha = 1/2$, and classifiy the critical $\vec{0}$ of the system as to type and stability;

(b) **Repeat** part (a) for $\alpha = 2$;

(c) In parts (a) and (b), we notice that solutions to the system exhibit two different behaviours near the critical point $\vec{0}$. **Find** the eigenvalues of the matrix $\begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix}$ in terms of α, and **determine** the value of α between 1/2 and 2 where the transition from one behaviour to other occurs.

(3) (8 points) **Find** the real-valued general solution to the following system of linear differential equations:

(a)

$$\frac{d\vec{y}}{dt} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \cdot \vec{y} + \begin{pmatrix} e^t \\ t \end{pmatrix};$$

(b)

$$\frac{d\vec{y}}{dt} = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \cdot \vec{y};$$

(c)

$$\frac{d\vec{y}}{dt} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \cdot \vec{y};$$

(d)

$$\frac{d\vec{y}}{dt} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \cdot \vec{y}.$$

(4) (3 points) Let A be a constant $n \times n$ matrix, and we consider the matrix valued differential equation $\Phi' = A \cdot \Phi$, with initial value $\Phi(t_0) = B$ for some invertible matrix B.

(a) Use the existence and uniqueness theorem for linear system of 1^{st} order differential equations to **show** that the above initial value problem has a unique solution defining on $\mathbb{R}$.

(b) Suppose we let $\Phi(t)$ to be the unique solution to the initial value $\Phi(0) = I$ (here I is the $n \times n$ identity matrix), use the uniqueness theorem to **show** that $\Phi(t) \cdot \Phi(s) = \Phi(t + s)$.

(c) **Show** that $\Phi(t) \cdot \Phi(-t) = I$, and hence **show** that $\Phi(t - s) = \Phi(t) \cdot \Phi(s)^{-1}$.

(5) (6 points) **Sketch** the phase portrait for each of the linear system of 1^{st} order differential equations:

(a) $\frac{d\vec{y}}{dt} = \begin{pmatrix} 1 & 1 \\ -5 & -3 \end{pmatrix} \cdot \vec{y};$

(b) $\frac{d\vec{y}}{dt} = \begin{pmatrix} -1 & 0 \\ -1 & -1/4 \end{pmatrix} \cdot \vec{y};$

(c) $\frac{d\vec{y}}{dt} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \cdot \vec{y}.$

(6) (6 points) For each of the following nonlinear system of 1^{st} order differential equations:

- **Find** all critical points of the above system and the corresponding linear system near each critical points;

- **Determine** the type and stability of the linear system associated to each critical points;

- **Draw** the phase portrait for the nonlinear system of differential equations.

(a)

$$\frac{dy_1}{dt} = (3 + y_1)(y_2 - y_1);$$
$$\frac{dy_2}{dt} = (4 - y_1)(y_2 + y_1).$$

(b)

$$\frac{dy_1}{dt} = y_1(1 - y_1 - y_2);$$
$$\frac{dy_2}{dt} = y_2(2 - y_1 - y_2).$$

(7) (4 points) Use Liapunov's function to **show** the stability of the following system of differential equations:

(a) For the system

$$\frac{dy_1}{dt} = -\frac{1}{2}y_1^3 + 2y_1y_2^2;$$
$$\frac{dy_2}{dt} = -2y_2^3,$$

show that $\vec{0}$ is an asymptotically stable critical point.

(b) For the system

$$\frac{dy_1}{dt} = 2y_1^3 - y_2^3;$$
$$\frac{dy_2}{dt} = 2y_1y_2^2 + 4y_1^2y_2 + 2y_2^3,$$

show that $\vec{0}$ is an unstable critical point.

End of Homework 4