MATH3230A Numerical Analysis

Tutorial 8 with solution

1 Recall:

1. **Polynomial interpolation**

   (a) **Chebyshev polynomials:**
   The Chebyshev polynomials are defined recursively as follows:

   $$T_0(x) = 1, \quad T_1(x) = x, \cdots, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

   In particular, for x in the interval $[-1, 1]$, the Chebyshev polynomials have the following closed forms for $n \geq 0$,

   $$T_n(x) = \cos(n \cos^{-1} x), \quad -1 \leq x \leq 1$$

   If the nodes x_i are chosen to be the roots of the Chebyshev polynomial T_{n+1}, then the error of the resulting interpolation polynomial for a given function $f(x)$ will be minimized and can be estimated by

   $$|f(x) - p(x)| \leq \frac{1}{2^n(n+1)!} \max_{|t| \leq 1} |f^{n+1}(t)|$$

   (b) **Hermite's interpolations:**
   Suppose we have the following observation data:

   $$\begin{pmatrix} f_0 = f(x_0) \\ f_0' = f'(x_0) \end{pmatrix}, \begin{pmatrix} f_1 = f(x_1) \\ f_1' = f'(x_1) \end{pmatrix}, \begin{pmatrix} f_2 = f(x_2) \\ f_2' = f'(x_2) \end{pmatrix}, \cdots, \begin{pmatrix} f_n = f(x_n) \\ f_n' = f'(x_n) \end{pmatrix},$$

   we would like to determine a polynomial $p(x)$ of degree $\leq 2n + 1$ such that

   $$p(x_i) = f(x_i), \quad p'(x_i) = f'(x_i), \quad , i = 0, 1, \cdots, n$$

   Define $u_i(x) = (1 - 2l_i'(x_i)(x - x_i))l_i^2(x), \quad v_i(x) = (x - x_i)l_i^2(x)$, where $l_i(x)$ are the Lagrange basis function. The Hermite's interpolation is

   $$H(x) = \sum_{i=0}^{n} f_i u_i(x) + \sum_{i=0}^{n} f_i' v_i(x) \tag{1}$$

   The following error estimate is true for $f \in C^{2n+2}[a, b]$ with $n + 1$ distinct interpolating points:

   $$f(x) - H(x) = \frac{f^{2n+2}(\xi)}{(2n+2)!}(x - x_0)^2(x - x_1)^2 \cdots (x - x_n)^2 \tag{2}$$

   An alternative method with Newton's interpolation can be used.The divided difference table is in a form uses $z_0, ..., z_{2n+1}$, specially $z_{2i} = z_{2i+1} = x_i$, and $f[z_{2i}, z_{2i+1}] = f'(z_{2i}) = f'(x_i)$. The rest is the same as usual.

2. **Numerical Integration:** We are going to estimate the integral

   $$\int_a^b f(x)dx$$

(a) **Trapezoidal Rule:**

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b))$$

The corresponding error is:

$$|\int_a^b f(x)dx - \frac{b-a}{2}(f(a)+f(b))| = \frac{1}{12}|f''(\zeta)|(b-a)^3 \leq \frac{K}{12}(b-a)^3$$

(b) **Composite Trapezoidal Rule:**

$$\int_a^b f(x)dx \approx h\left(\frac{f(x_0)}{2} + f(x_1) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right)$$

The corresponding error is:

$$|E_h(f)| = \frac{b-a}{12}|f''(\eta)|h^2$$

2 Exercises:

Please submit solutions of problems with star(*) before 6:30PM on Wednesday and finish the rest by yourself.

1. (a) Let

$$U_{n+1}(x) = \frac{\sin\left((n+2)\,\theta\right)}{\sin(\theta)}, \quad x = \cos(\theta)$$

i. Show that

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

ii. Prove that

$$T_n'(x) = nU_{n-1}(x),$$

where $T_n(x) = \cos(n\cos^{-1}(x))$.

(b) Consider the Chebyshev polynomial

$$T_n(x) = \cos(n\cos^{-1}(x)), \quad x \in [-1,1].$$

i. Let p be a monic polynomial of degree n. Show that

$$\max_{-1\leq x\leq 1} |p(x)| \geq 2^{1-n}.$$

ii. Let f be a smooth function, $\{x_i\}_{i=0}^n$ the roots of Chebyshev polynomials T_{n+1}, and p the interpolation polynomial of f at nodes $\{x_i\}_{i=0}^n$. Show that

$$|f(x) - p(x)| \leq \frac{1}{2^n(n+1)!}\max_{|t|\leq 1}|f^{(n+1)}(t)|, \quad x \in [-1,1].$$

Solution.

(a) i.

$$U_0(x) = \frac{\sin(\theta)}{\sin(\theta)} = 1$$

$$U_1(x) = \frac{\sin(2\theta)}{\sin(\theta)} = 2\cos(\theta) = 2x.$$

$$\begin{aligned} U_{n+1}(x) &= \frac{\sin((n+2)\theta)}{\sin(\theta)} = \frac{\sin((n+2)\theta) + \sin(n\theta) - \sin(n\theta)}{\sin(\theta)} \\ &= \frac{2\sin((n+1)\theta)\cos(\theta) - \sin(n\theta)}{\sin(\theta)} \\ &= 2xU_n(x) - U_{n-1}(x) \end{aligned}$$

ii. Since $T_n(x) = \cos(n\cos^{-1}(x))$,
we have

$$\begin{aligned} T_n'(x) &= -\sin(n\cos^{-1}(x)) \cdot n \cdot \frac{-1}{\sqrt{1-x^2}} \\ &= \frac{n\sin(n\cos^{-1}(x))}{\sqrt{1-x^2}} \\ &= \frac{n\sin(n\theta)}{\sin(\theta)} \\ &= nU_{n-1}(x) \end{aligned}$$

(b) i. Please find the proof in Lecture notes.

ii. By the interpolation error estimate, we have

$$\max_{|x|\le 1} |f(x) - p(x)| \le \frac{1}{(n+1)!} \max_{|x|\le 1} |f^{n+1}(x)| \max_{|x|\le 1} |(x-x_0)(x-x_1)\cdots(x-x_n)|$$

By the property of the monic function in part (i), we have

$$\max_{|x|\le 1} |(x-x_0)(x-x_1)\cdots(x-x_n)| \ge 2^{-n}$$

This minimum value is attained if $(x-x_0)(x-x_1)\cdots(x-x_n)$ is the monic multiple of T_{n+1}, i.e., $2^{-n}T_{n+1}$. The nodes are the roots of T_{n+1}, namely

$$x_i = \cos\left(\frac{2i-1}{2n+2}\pi\right), \quad i = 1, 2, \cdots n+1.$$

Combining the above results, we have

$$|f(x) - p(x)| \le \frac{1}{2^n(n+1)!} \max_{|t|\le 1} |f^{(n+1)}(t)|, \quad x \in [-1, 1]$$

□

2. (a) For the following set of data,

x	0	1	2
$f(x)$	1	3	0
$f'(x)$	0	1	0

compute a Hermite's interpolation (in either Lagrange form or in Newton form).

(b) Suppose $f \in C^{2n+2}[a, b]$ and $H(x)$ is its Hermite's interpolation at the $n+1$ distinct points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Write down the error estimate of Hermite's interpolation, and point out for what polynomials the Hermite's interpolation is exact.

Solution. (a) In Lagrange form:

$$H(x) = \frac{1}{4}(1+3x)(x-1)^2(x-2)^2 + 3x^2(x-2)^2 + x^2(x-1)^2(x-2)^2$$

In Newton form:

$$H(x) = 1 + 2x^2 - 3x^2(x-1) + \frac{3}{4}x^2(x-1)^2 + \frac{7}{4}x^2(x-1)^2(x-2)$$

(b)

$$f(x) - H(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}(x-x_0)^2(x-x_1)^2\cdots(x-x_n)^2$$

The interpolation on $n+1$ distinct points is exact for any polynomial of degree $k \le 2n+1$.

□

3. Let us consider a real function f and its integral over $[a,b]$:

$$\int_a^b f(x)dx \tag{3}$$

   (a) Write down the trapezoidal rule to approximate the integral (3).
   (b) Write down the error estimate of the trapezoidal rule in (a).
   (c) Use the trapezoidal rule to compute the integral

$$\int_0^2 e^{-x^2}dx.$$

   (d) Let us consider a partition of the interval $[a,b]$ with n equally-spaced smaller subintervals using the nodal points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\,,$$

   where $h = (b-a)/n$ is the length of each interval.
      i. Please write down the composition trapezoidal rule approximate the integral (3).
      ii. Assume that $f \in C^2[a,b]$. Derive the error estimate for the composition trapezoidal rule.
      iii. Consider the integral :

$$\int_0^\pi \sin x dx. \tag{4}$$

      How large must n be if the error of the composite trapezoidal rule for computing integral (4) is not bigger than 10^{-8}.

Proof. (a) Trapezoidal rule is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b)),$$

(b) The error of Trapezoidal rule is

$$\frac{1}{12}K(b-a)^3 \quad \text{where } \max_{x\in[a,b]} |f'(x)| \le K$$

(c) By the Trapezoidal rule,

$$\int_0^2 e^{-x^2}dx \approx \frac{2}{2}[e^0 + e^{-4}] \approx 1.0183$$

(d) i. Composite trapezoidal rule is

$$\int_a^b f(x)dx \approx h(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}).$$

ii. Using the error estimate for the trapezoidal rule,

$$\begin{aligned}
&E_n(f)\\
&= \sum_{i=1}^n \left\{ \int_{x_{i-1}}^{x_i} f(x)dx - \frac{x_i - x_{i-1}}{2}\left(f(x_{i-1}) + f(x_i)\right) \right\}\\
&= \sum_{i=1}^n -\frac{1}{12} f''(\xi_i)(x_i - x_{i-1})^3\\
&= \sum_{i=1}^n -\frac{1}{12} f''(\xi_i) h^3\\
&= -\frac{h^3}{12}\sum_{i=1}^n f''(\xi_i).
\end{aligned} \tag{5}$$

Using the mean-value theorem, there exists a point $\eta \in [a, b]$ such that

$$f''(\eta) = \frac{1}{n} \sum_{i=1}^{n} f''(\xi_i).$$

The error estimate for the composite trapezoidal rule:

$$E_h(f) = -\frac{nh^3}{12} f''(\eta) = -\frac{b-a}{12} f''(\eta) h^2.$$

That is

$$|E_h(f)| \leq \frac{K}{12}(b-a)h^2 \quad \text{where } \max_{x \in [a,b]} |f''(x)| \leq K$$

iii. The error of the composite trapezoidal rule is

$$E_h(f) = -\frac{b-a}{12} f''(\eta) h^2 = -\frac{(b-a)^3}{12n^2} f''(\eta) = -\frac{\pi^3}{12n^2} f''(\eta)$$

and $f'(x) = \cos x$, $f''(x) = -\sin x$, so we have $|f''(x)| \leq 1$, then

$$|E_h(f)| \leq \frac{\pi^3}{12n^2} \leq 10^{-8},$$

this shows $n > 16074$.

□