MATH3230A Numerical Analysis

Tutorial 5 with solution

1 Recall:

1. LU **factorization with partial pivoting:**

   Idea:
   Permute the rows so that the largest entry in magnitude in the column becomes the pivot.

   Strategy:
   At the kth stage of the LU factorization, suppose the matrix A becomes $A_k = (a_{ij}^{(k)})$. Then determine an index p_k for which $|a_{p_k,k}^{(k)}|$ is the largest among all $|a_{j,k}^{(k)}|$ for $k \le j \le n$. Then interchange rows k and p_k before proceeding the next step of the factorization.

2. **Least-square solution for general non-square linear systems:**
   Let A be a $m \times n$ matrix, with $m > n$ and now we consider a general linear system

   $$Ax = b. \tag{1}$$

   The least square solution seeks for some vector x that minimizes the error $(Ax - b)$ in the least square sense, that is

   $$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2$$

   We assume that the columns of A are linearly independent. We define

   $$f(x) = \|Ax - b\|_2^2$$

   The minimizer of $f(x)$ satisfies the normal equation

   $$A^T Ax = A^T b.$$

3. **Newton's method for general nonlinear systems** Consider the following system of nonlinear equation:

   $$\begin{aligned} f_1(x_1, x_2, \cdots, x_n) &= 0, \\ f_2(x_1, x_2, \cdots, x_n) &= 0, \\ &\vdots \\ f_n(x_1, x_2, \cdots, x_n) &= 0. \end{aligned} \tag{2}$$

   where each $f_i(x_1, x_2, \cdots, x_n)$ is a nonlinear function of n variables $x_1, x_2, \cdots, x_n$. The system can be written simply as

   $$F(\mathbf{x}) = 0,$$

   where $F(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_n(\mathbf{x}))^T$, and $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$.
   The Newton's method for solving $F(\mathbf{x}) = 0$ is as follows: Given $\mathbf{x_0}$, for $k = 0, 1, 2, \cdots$, do the following

   (a) Compute $\mathbf{d_k}$ by solving

   $$F'(\mathbf{x_k})\mathbf{d_k} = -F(\mathbf{x_k}).$$

(b) Update $\mathbf{x_{k+1}}$ by

$$\mathbf{x_{k+1}} = \mathbf{x_k} + \mathbf{d_k}.$$

4. **Convergence of Newton's method for general nonlinear systems:**
Assume F satisfies the following assumptions:

(a) There is a solution $\mathbf{x}^*$ to the equation $F(\mathbf{x}) = \mathbf{0}$;

(b) Jacobian matrix $F'(\mathbf{x}^*)$ is nonsingular;

(c) Jacobian $F' : \Omega \to \mathbb{R}^{n\times n}$ is Lipschitz continuous with Lipschitz constant γ, i.e. we have

$$\|F'(\mathbf{x}) - F'(\mathbf{x}^*)\| \leq \gamma\|\mathbf{x} - \mathbf{x}^*\|$$

Theorem 1 (Quadratically convergence of Newton's method)**.** *Under the three assumptions above, there exist constant $K > 0$ and $\delta > 0$ such that for any $\mathbf{x_0} \in B_\delta(\mathbf{x}^*)$, the sequence $\{\mathbf{x_n}\}$ generated by the Newton's method satisfies $\mathbf{x_n} \in B_\delta(\mathbf{x}^*)$, and*

$$\|\mathbf{x_{n+1}} - \mathbf{x}^*\| \leq K\|\mathbf{x_n} - \mathbf{x}^*\|^2, \quad n = 0, 1, 2, \cdots$$

So Newton's method converges quadratically.

2 Exercises:

Please submit solutions of problems with star(*) before 6:30PM on Wednesday and finish the rest by yourself.

1. * Suppose $A \in \mathbb{R}^{n\times n}$ is strictly column diagonally dominant, which means that for each k,

$$|a_{kk}| > \sum_{j\neq k} |a_{jk}|$$

Show that if LU factorization with partial pivoting is applied to A, no row interchanges take place.

Solution.

Since $|a_{11}| > \sum_{j\neq 1} |a_{j1}|$, there is no need to interchange row for the first step. Then we do the Gaussian elimination. We denote the $A^{(1)}$ be the matrix A after the first step Gaussian elimination. Then we obtain for $j > 1$, $a^{(1)}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}$. Hence, $\forall k > 1$,

$$\begin{aligned}
\sum_{\substack{j=2\\j\neq k}}^{n} \left|{a^{(1)}}_{j,k}\right| &= \sum_{\substack{j=2\\j\neq k}}^{n} \left|a_{j,k} - a_{j,1}\frac{a_{1,k}}{a_{1,1}}\right| \\
&\leqslant \sum_{\substack{j=2\\j\neq k}}^{n} |a_{j,k}| + \sum_{\substack{j=2\\j\neq k}}^{n} |a_{j,1}| \frac{|a_{1,k}|}{|a_{1,1}|} \\
&< |a_{k,k}| - |a_{1,k}| + |a_{1,k}| \sum_{\substack{j=2\\j\neq k}}^{n} \frac{|a_{j,1}|}{|a_{1,1}|} \\
&\leqslant |a_{k,k}| - |a_{1,k}| + |a_{1,k}| \left(1 - \frac{|a_{k,1}|}{|a_{1,1}|}\right) \\
&\leqslant |a_{k,k}| - |a_{1,k}| \frac{|a_{k,1}|}{|a_{1,1}|} \\
&\leqslant \left|a_{k,k} - a_{1,k}\frac{a_{k,1}}{a_{1,1}}\right| \\
&\leqslant \left|{a^{(1)}}_{k,k}\right|
\end{aligned}$$

Therefore $A^{(1)}$ is also strictly column diagonally dominant matrix. The argument repeats and the result is proved.

□

2. (a) Assume that if matrix $A_{m\times n}$ is full rank, we can find the least square solution x^* which minimizes the energy:

$$\frac{1}{2}\left\|Ax-b\right\|_2^2.$$

Prove that solving for x^* is equal to solve the following equation

$$A^TAx = A^Tb.$$

(b) * Solve the following linear system using the Cholesky factorization in least square sense.

$$Ax=b, \quad A=\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad b=\begin{pmatrix} 2 \\ 1 \\ 1 \\ -2 \end{pmatrix}$$

Solution. (a) See lecture notes.

(b) Note that

$$A^TA=\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

Then

$$A^TA=\begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}^T \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} = R^TR$$

First we solve $Ry=A^Tb$, we have

$$y=\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{bmatrix}$$

then we solve $Rx=y$, we have

$$x=\begin{bmatrix} \frac{3}{2} \\ 1 \\ \frac{3}{2} \end{bmatrix}$$

□

3. * Consider the following system of linear equation

$$\begin{aligned} x \quad\quad\quad - 3z &= 2 \\ 2x - 2y + z &= 1 \\ - y + 3z &= -2 \end{aligned}$$

(a) Formulate the Newton's method for solving the above system of linear equation.

(b) Find the number of iteration for the Newton's method to return the exact solution with initial guess $X_0 = (0,0,0)^T$.

(c) What does Newton's method reduce to for the linear system $Ax = b$ given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

where A is a nonsingular matrix?

Solution. (a) Let

$$\begin{aligned} f_1(x,y,z) &= x - 3z - 2 \\ f_2(x,y,z) &= 2x - 2y + z - 1 \\ f_3(x,y,z) &= -y + 3z + 2. \end{aligned}$$

Denote $X = (x,y,z)^T$, $F(X) = (f_1(X), f_2(X), f_3(X))^T$.
Note that

$$F'(X) = \begin{pmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 0 & -1 & 3 \end{pmatrix} := A$$

Set initial value $X_0 = (x_0, y_0, z_0)^T$, Newton's method gives:

$$X_{k+1} = X_k - F'(X_k)^{-1}F(X_k) = X_k - A^{-1}F(X_k)$$

(b) The first iteration of Newton's method will be

$$X_1 = X_0 - A^{-1}F(X_0) = (5,5,1)^T,$$

which is the solution of the system of linear equation. So one iteration will return the exact solution.

(c) Since $f_j(x_1, \cdots, x_n) = a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n - b_j$, we have $\frac{\partial f_j}{\partial x_i} = a_{ji}$. Hence,

$$F'(X) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} := A.$$

Further,

$$\begin{aligned} F(X_0) &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} X_0 - \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\ &= AX_0 - b \end{aligned}$$

Thus, given X_0, we have

$$\begin{aligned} X_1 &= X_0 - A^{-1}(AX_0 - b) \\ &= X_0 - A^{-1}AX_0 + A^{-1}b \\ &= A^{-1}b \end{aligned}$$

So given any X_0, the solution to the linear system is X_1.

□

4. Consider the following system of nonlinear equations

$$F(\mathbf{x}) = 0, \tag{3}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function.

(a) Please write down the three standard assumptions on F such that the Newton's method works and converges quadratically.

(b) * Let $\mathbf{x}^*$ denote the solution that $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$. Consider the function

$$\mathbf{g}(t) = \mathbf{F}(\mathbf{x}^* + (\mathbf{x} - \mathbf{x}^*)t), \quad t \in [0, 1],$$

show that

$$\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}^*) = \int_0^1 \mathbf{F}'(\mathbf{x}^* + (\mathbf{x} - \mathbf{x}^*)t)(\mathbf{x} - \mathbf{x}^*)dt.$$

(c) Please prove that: there exists a $\delta > 0$ such that for all $\mathbf{x} \in B_\delta(\mathbf{x}^*)$, it holds true that:

i. $\|\mathbf{F}'(\mathbf{x}) - \mathbf{F}'(\mathbf{x}^*)\| \le \gamma \|\mathbf{x} - \mathbf{x}^*\|$,

ii. $\|\mathbf{F}'(\mathbf{x})\| \le 2\|\mathbf{F}'(\mathbf{x}^*)\|$,

iii. $\|\mathbf{F}'(\mathbf{x})^{-1}\| \le 2\|\mathbf{F}'(\mathbf{x}^*)^{-1}\|$

iv. $\frac{1}{2}\|\mathbf{F}'(\mathbf{x}^*)^{-1}\|^{-1}\|\mathbf{x} - \mathbf{x}^*\| \le \|\mathbf{F}(\mathbf{x})\| \le 2\|\mathbf{F}'(\mathbf{x}^*)\|\|\mathbf{x} - \mathbf{x}^*\|$,

where γ is the Lipschitz constant of the Jacobian F'.

(d) * Based on the the results from (b)(c) and your assumptions in (a), please prove the quadratic convergence of the Newton's method for system of nonlinear equations.

Solution. (a) The assumptions are:

1. There is a solution x^* to the equation $F(x) = 0$.
2. Jacobian matrix $F'(x^*)$ is nonsingular.
3. Jacobian $F' : \Omega \to R^{n \times n}$ is Lipschitz continuous.

(b) By the fundamental theorem of calculus and the change of variables,

$$g(1) - g(0) = \int_0^1 \frac{dg}{dt} dt$$

Substituting $g(t) = \mathbf{F}(\mathbf{x}^* + (\mathbf{x} - \mathbf{x}^*)t$, we have

$$\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}^*) = \int_0^1 \mathbf{F}'(\mathbf{x}^* + (\mathbf{x} - \mathbf{x}^*)t)(\mathbf{x} - \mathbf{x}^*)dt$$

(c) Part of (iv): Note that if $\mathbf{x} \in B_\delta(\mathbf{x}^*)$ then $\mathbf{x}^* + t(\mathbf{x} - \mathbf{x}^*) \in B_\delta(\mathbf{x}^*)$ for all $0 \le t \le 1$. Using the result from (b) and inequality (ii), we can easily obtain the right side of (iv):

$$\|F(\mathbf{x})\| \le \int_0^1 \|F'(\mathbf{x}^* + t(\mathbf{x} - \mathbf{x}^*))\| \; \|\mathbf{x} - \mathbf{x}^*\| dt \le 2\|F'(\mathbf{x}^*)\|\|\mathbf{x} - \mathbf{x}^*\|$$

For the left side, please refer to page 106 in the lecture notes.
Also, please read the proof of (i)-(iii) in the notes.

(d) By the definition we have

$$\begin{aligned}
\mathbf{x_{n+1}} - \mathbf{x}^* &= \mathbf{x_n} - \mathbf{x}^* - \mathbf{F}'(\mathbf{x_n})^{-\mathbf{1}}\mathbf{F}(\mathbf{x_n}) \\
&= \mathbf{F}'(\mathbf{x_n})^{-\mathbf{1}} \left(\mathbf{F}'(\mathbf{x_n})(\mathbf{x_n} - \mathbf{x}^*) - \mathbf{F}(\mathbf{x_n})\right) \\
&= \mathbf{F}'(\mathbf{x_n})^{-\mathbf{1}} \left(\mathbf{F}'(\mathbf{x_n})(\mathbf{x_n} - \mathbf{x}^*) - \int_0^1 \mathbf{F}'(\mathbf{x}^* + (\mathbf{x_n} - \mathbf{x}^*)t)(\mathbf{x_n} - \mathbf{x}^*)dt\right) \\
&= \mathbf{F}'(\mathbf{x_n})^{-\mathbf{1}} \int_0^1 \left(\mathbf{F}'(\mathbf{x_n}) - \mathbf{F}'(\mathbf{x}^* + (\mathbf{x_n} - \mathbf{x}^*)t)\right)(\mathbf{x_n} - \mathbf{x}^*)dt
\end{aligned}$$

Using the properties, we have

$$\|\mathbf{x_{n+1}} - \mathbf{x}^*\| \leq (2\|F'(\mathbf{x}^*)^{-1}\|)\gamma\|\mathbf{x_n} - \mathbf{x}^*\|^2/2$$

This completes the proof.

□

5. Let A be nonsingular, and let LU factorization with pivoting takes the form:

$$U = L_{n-1}P_{n-1} \cdots L_2P_2L_1P_1A$$

where L_i is elementary matrix and U is upper triangular matrix.

(a) * Solve the following system using the LU factorization with pivoting, and please find each of the matrix

$$U, L_i, P_i, \qquad i = 1, 2, \cdots, n-1,$$

where n is the row number of the matrix.

$$\begin{pmatrix} 3 & 6 & 2 & 5 & 1 \\ 5 & 5 & 2 & 4 & 4 \\ 6 & 6 & 8 & 2 & 2 \\ 4 & 7 & 9 & 3 & 8 \\ 5 & 3 & 5 & 7 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 46 \\ 57 \\ 60 \\ 97 \\ 64 \end{pmatrix}$$

(b) Denote $P = P_{n-1} \cdots P_1$. Prove

$$PA = LU$$

where L is lower triangular matrix with 1 as its diagonal entries and all entries of L satisfy

$$|l_{ij}| \leq 1,$$

U is upper triangular matrix.

(c) Find P, L, U in (b) according to question (a) such that

$$PA = LU$$

where A is the matrix in (a).

(d) Explain when it is necessary to do the LU factorization with partial pivoting.

Solution.

(a) $R3 \leftrightarrow R1$

$$\begin{pmatrix} & & 1 & & \\ & 1 & & & \\ 1 & & & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 3 & 6 & 2 & 5 & 1 \\ 5 & 5 & 2 & 4 & 4 \\ 6 & 6 & 8 & 2 & 2 \\ 4 & 7 & 9 & 3 & 8 \\ 5 & 3 & 5 & 7 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ 5 & 5 & 2 & 4 & 4 \\ 3 & 6 & 2 & 5 & 1 \\ 4 & 7 & 9 & 3 & 8 \\ 5 & 3 & 5 & 7 & 2 \end{pmatrix}$$

LU

$$\begin{pmatrix} 1 & & & & \\ -\frac{5}{6} & 1 & & & \\ -\frac{1}{2} & & 1 & & \\ -\frac{2}{3} & & & 1 & \\ -\frac{5}{6} & & & & 1 \end{pmatrix}\begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ 5 & 5 & 2 & 4 & 4 \\ 3 & 6 & 2 & 5 & 1 \\ 4 & 7 & 9 & 3 & 8 \\ 5 & 3 & 5 & 7 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 0 & -\frac{14}{3} & \frac{7}{3} & \frac{7}{3} \\ & 3 & -2 & 4 & 0 \\ & 3 & \frac{11}{3} & \frac{5}{3} & \frac{20}{3} \\ & -2 & -\frac{5}{3} & \frac{16}{3} & \frac{1}{3} \end{pmatrix}$$

$R3 \leftrightarrow R2$

$$\begin{pmatrix} 1 & & & & \\ & & 1 & & \\ & 1 & & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 0 & -\frac{14}{3} & \frac{7}{3} & \frac{7}{3} \\ & 3 & -2 & 4 & 0 \\ & 3 & \frac{11}{3} & \frac{5}{3} & \frac{20}{3} \\ & -2 & -\frac{5}{3} & \frac{16}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & 0 & -\frac{14}{3} & \frac{7}{3} & \frac{7}{3} \\ & 3 & \frac{11}{3} & \frac{5}{3} & \frac{20}{3} \\ & -2 & -\frac{5}{3} & \frac{16}{3} & \frac{1}{3} \end{pmatrix}$$

LU

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & 0 & 1 & & \\ & -1 & & 1 & \\ & \frac{2}{3} & & & 1 \end{pmatrix}\begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & 0 & -\frac{14}{3} & \frac{7}{3} & \frac{7}{3} \\ & 3 & \frac{11}{3} & \frac{5}{3} & \frac{20}{3} \\ & -2 & -\frac{5}{3} & \frac{16}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & & -\frac{14}{3} & \frac{7}{3} & \frac{7}{3} \\ & & \frac{17}{3} & -\frac{7}{3} & \frac{20}{3} \\ & & -3 & 8 & \frac{1}{3} \end{pmatrix}$$

$R4 \leftrightarrow R3$

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & & 1 & \\ & & 1 & & \\ & & & & 1 \end{pmatrix}\begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & & -\frac{14}{3} & \frac{7}{3} & \frac{7}{3} \\ & & \frac{17}{3} & -\frac{7}{3} & \frac{20}{3} \\ & & -3 & 8 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & & \frac{17}{3} & -\frac{7}{3} & \frac{20}{3} \\ & & -\frac{14}{3} & \frac{7}{3} & \frac{7}{3} \\ & & -3 & 8 & \frac{1}{3} \end{pmatrix}$$

LU

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & \frac{14}{17} & 1 & \\ & & \frac{9}{17} & & 1 \end{pmatrix}\begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & & \frac{17}{3} & -\frac{7}{3} & \frac{20}{3} \\ & & -\frac{14}{3} & \frac{7}{3} & \frac{7}{3} \\ & & -3 & 8 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & & \frac{17}{3} & -\frac{7}{3} & \frac{20}{3} \\ & & & \frac{3}{17} & \frac{133}{17} \\ & & & \frac{115}{17} & \frac{197}{51} \end{pmatrix}$$

$R5 \leftrightarrow R4$

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & & 1 \\ & & & 1 & \end{pmatrix}\begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & & \frac{17}{3} & -\frac{7}{3} & \frac{20}{3} \\ & & & \frac{3}{17} & \frac{133}{17} \\ & & & \frac{115}{17} & \frac{197}{51} \end{pmatrix} = \begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & & \frac{17}{3} & -\frac{7}{3} & \frac{20}{3} \\ & & & \frac{115}{17} & \frac{197}{51} \\ & & & \frac{3}{17} & \frac{133}{17} \end{pmatrix}$$

LU

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & -\frac{7}{115} & 1 \end{pmatrix}\begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & & \frac{17}{3} & -\frac{7}{3} & \frac{20}{3} \\ & & & \frac{115}{17} & \frac{197}{51} \\ & & & \frac{3}{17} & \frac{133}{17} \end{pmatrix} = \begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & & \frac{17}{3} & -\frac{7}{3} & \frac{20}{3} \\ & & & \frac{115}{17} & \frac{197}{51} \\ & & & & \frac{2618}{345} \end{pmatrix}$$

(b) From pivoting form:

$$U = L_{n-1}P_{n-1}...L_2P_2L_1P_1A$$

Now we claim that there exists some $L^{(k)}$ such that

$$L_{n-1}P_{n-1}...L_2P_2L_1P_1 = L^{(n-1)}L^{(n-2)}\dots L^{(1)}P_{n-1}P_{n-2}\dots P_1$$

such that the structure of $L^{(k)}$ is equal to L_k but with the subdiagonal entries permuted. We prove the claim as follows: Define

$$L^{(n-1)} = L_{n-1}, \quad L^{(n-2)} = P_{n-1}L_{n-2}P_{n-1}^{-1},$$

$$L^{(n-3)} = P_{n-1}P_{n-2}L_{n-3}P_{n-2}^{-1}P_{n-1}^{-1}\ , \dots$$

we have

$$\begin{aligned} & L^{(n-1)}L^{(n-2)}\dots L^{(1)}P_{n-1}P_{n-2}\dots P_1 \\ =& L_{n-1}(P_{n-1}L_{n-2}P_{n-1}^{-1})(P_{n-1}P_{n-2}L_{n-3}P_{n-2}^{-1}P_{n-1}^{-1})\dots P_{n-1}P_{n-2}\dots P_1 \\ =& L_{n-1}P_{n-1}L_{n-2}P_{n-2}\dots L_1P_1 \end{aligned} \tag{4}$$

Therefore we have

$$U = (L^{(n-1)}L^{(n-2)}\dots L^{(1)})(P_{n-1}P_{n-2}\dots P_1)A$$

As $L^{(k)}$ has the same structure as L_k for all k, we can combine them to form a lower triangular matrix with 1's as diagonal entries. Therefore we have proved the result.

(c) P and U are the same as in (a).

$$L = \begin{pmatrix} 1 & & & & \\ \frac{1}{2} & 1 & & & \\ \frac{2}{3} & 1 & 1 & & \\ \frac{5}{6} & -\frac{2}{3} & -\frac{9}{17} & 1 & \\ \frac{5}{6} & 0 & -\frac{14}{17} & \frac{7}{115} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 6 & 6 & 8 & 2 & 2 \\ & 3 & -2 & 4 & 0 \\ & & \frac{17}{3} & -\frac{7}{3} & \frac{20}{3} \\ & & & \frac{115}{17} & \frac{197}{51} \\ & & & & \frac{2618}{345} \end{pmatrix}, \quad P = \begin{pmatrix} & & 1 & & \\ 1 & & & & \\ & & & 1 & \\ & & & & 1 \\ & 1 & & & \end{pmatrix}$$

(d) When the absolute value of diagonal entry is too small at certain stage of the LU factorization, it may force algorithm to stop or cause buffer overflow.

□