MATH3230A Numerical Analysis

Tutorial 2 with solution

1 Recall:

1. **Quasi-Newton method:**
   Recall the Newton's method
   $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \dots \tag{1}$$
   we see that at each iteration, we need to compute the derivative $f'(x_k)$. This might be a big trouble for some cases. For instance
   (a) The expression $f(x)$ is unknown;
   (b) The derivative $f'(x)$ is very expensive to compute;
   (c) The value of function f may be the result of a long numerical calculation, so the derivative has no formula available.

2. **Constant slope method:** By approximating $f'(x_k)$ by a constant $g_k = g$, the Newton's method becomes
   $$x_{k+1} = x_k - \frac{f(x_k)}{g}, \quad k = 0, 1, 2, \dots \tag{2}$$
   This is called the constant slope method. In particular, we might take $g = f'(x_0)$.

3. **Fixed-point iterative methods:** Both Newton's method and the Quasi-Newton's method can be seen as some special case of fixed-point iterative methods. For a given function $\varphi(x)$, x^* is called its fixed point if x^* satisfies
   $$\varphi(x^*) = x^*.$$
   The iterative method
   $$x_{k+1} = \varphi(x_k), \quad k = 0, 1, 2, \dots$$
   is called a fixed-point iteration associated with the function $\varphi(x)$, and $\varphi(x)$ is called the iterative function.

   Theorem 1 (Convergence of fixed-point iterative method)**.** *If the iterative function $\varphi(x)$ satisfies the condition*
   $$|\varphi'(x^*)| < 1,$$
   then there exists a $\delta > 0$ such that for any $x_0 \in [x^* - \delta, x^* + \delta]$, the fixed-point iteration converges. If $\varphi'(x^*) \neq 0$, the convergence is linear with convergence rate $\rho = |\varphi'(x^*)|$. If
   $$\varphi'(x^*) = \varphi''(x^*) = \cdots = \varphi^{(p-1)}(x^*) = 0, \quad \textit{but} \quad \varphi^{(p)} \neq 0,$$
   then the fixed-point interation converges with order p.

4. **Cases with multiple zeros:** A point x^* is called a zero of the function $f(x)$ with multiplicity $m \geq 1$ if it holds
   $$f(x^*) = f'(x^*) = \cdots = f^{(m-1)}(x^*) = 0,$$
   but $f^{(m)}(x^*) \neq 0$.

5. **Convergence of Newton's method in the case of multiplicity:** We have the following local convergence of the Newton's method when it is applied for solving a nonlinear equation with a zero of multiplicity m:
   (a) It converges quadratically for $m = 1$, namely when x^* is a single zero;
   (b) It converges only linearly to the multiple zero x^* with rate $\rho = 1 - \frac{1}{m}$ for $m > 1$. If $m = 2$, the convergence rate is $1/2$, the same as the convergence rate of the bisection algorithm. The Newton's method converges more slowly when m is larger.

2 Exercises:

Please do the star problem (*) in tutorial class and finish the rest after class.

1. *

   (a) Show that the sequence $x_n = 10^{-2^n}$ with initial guess $x_0 = \frac{1}{10}$ converges quadratically to 0.

   (b) Show that the sequence $x_n = 10^{-n^k}$ with initial guess $x_0 = 1$ does not converge to 0 quadratically if k is a positive integer and $k > 1$.

   Solution.

   (a) Clearly, $x_n = 10^{-2^n} \to 0$ as $n \to \infty$. Then, we have

   $$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1.$$

   Thus the sequence converges quadratically to 0.

   (b) When k is a positive integer and $k > 1$,

   $$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^2} = \lim_{n\to\infty} \frac{10^{-(n+1)^k}}{10^{-2n^k}} = \lim_{n\to\infty} 10^{2n^k-(n+1)^k}.$$

   Note that $2n^k - (n+1)^k = 2n^k - n^k - a_{k-1}n^{k-1} - \cdots - 1 = n^k - q_{k-1}(n)$, where $q_{k-1}(n)$ is a $k-1$ degree polynomial in n. Hence,

   $$\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|^2} = \lim_{n\to\infty} 10^{2n^k-(n+1)^k} = \lim_{n\to\infty} 10^{n^k-q_{k-1}(n)} = \infty.$$

   Therefore, the sequence does not converge to 0 quadratically.

   □

2. * Please read the Lecture Notes for the bisection method and answer the questions.
   Consider the following nonlinear equation:

   $$f(x) := 2x^3 + 4x^2 - 5 = 0 \tag{3}$$

   and use $I = [0, 1]$ as the initial interval.

   (a) Find the minimum number of iterations required to approximate the solution with an absolute error of less than 10^{-5}.

   (b) Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the bisection method. Please calculus the value of x_0, x_1 and x_2.

   Solution.

   (a) $|x_n - x^*| \leq 2^{-(n+1)}(1-0) < 10^{-5}$. Since $2^{-17} < 10^{-5}$, the iteration $n = 16$.

   (b)

   $$\begin{aligned} x_0 &= \frac{1}{2}(a_0 + b_0) = \frac{1}{2}. \\ f(x_0) < 0 &\Rightarrow a_1 = x_0,\ b_1 = b_0 = 1 \\ x_1 &= \frac{1}{2}(a_1 + b_1) = \frac{3}{4}. \\ f(x_1) < 0 &\Rightarrow a_2 = \frac{3}{4},\ b_2 = 1 \\ x_2 &= \frac{1}{2}(a_2 + b_2) = \frac{7}{8}. \end{aligned}$$

   □

3. * Consider the following nonlinear equation

$$f(x) := x^3 - 2x^2 + x = 0,$$

and the following iterative sequence

$$\begin{cases} x_{n+1} = \phi(x_n), \\ x_0 \text{ is given,} \end{cases}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an iteration function.

(a) Consider the iterative function

$$\phi(x) := x - 2\frac{f(x)}{f'(x)}.$$

Determine if the fixed-point iteration has local convergence near $x = 1$. If so, find the order of convergence.

(b) Consider the iterative function

$$\phi(x) := x + 10\frac{f(x)}{f'(x)}.$$

Determine if the fixed-point iteration has local convergence near $x = 1$. If so, find the order of convergence.

(c) Consider the iterative function

$$\varphi(x) = x - \frac{hf(x)}{f(x+h) - f(x)}$$

where $h > 0$ is a parameter to be chosen. Find the range of h such that the fixed-point iteration has local convergence near $x = 0$, if there is any.

(d) Suggest an iterative function $\varphi(x)$ and the resulting new fixed-point method, which is different from the methods mentioned above, to solve the equation.

Solution. Note that

$$\begin{aligned} f &= x(x-1)^2, \\ f' &= (x-1)(3x-1), \\ f'' &= 6x - 4, \\ f''' &= 6. \end{aligned}$$

(a) Consider

$$\begin{aligned} \phi'(x) &= 1 - 2\frac{[f']^2 - f''f}{[f']^2} = -1 + \frac{2f''f}{(f')^2} \\ &= -1 + \frac{2x(x-1)^2(6x-4)}{(x-1)^2(3x-1)^2} \\ &= -1 + \frac{4x(3x-2)}{(3x-1)^2}. \end{aligned}$$

Hence, we have $\phi'(1) = 0$.
Also consider that

$$\phi''(x) = \frac{4}{(3x-1)^4}\big(2(3x-1)^3 - 6(3x-1)(3x-2)x\big) \tag{4}$$

$$= \frac{8}{(3x-1)^3}. \tag{5}$$

Hence, $\phi''(1) \neq 0$. We conclude that the fixed-point iteration has convergence of order 2.

(b) Note that

$$\begin{aligned}\phi'(x) &= 1 + 10\frac{[f']^2 - f''f}{[f']^2} = 11 - 10\frac{f''f}{(f')^2} \quad (6)\\ &= 11 - 10\frac{2x(3x-2)}{(3x-1)^2}. \quad (7)\end{aligned}$$

Because $\phi'(1) = 11 - 5 = 6 > 1$. this iteration can not ensure local convergence near $x = 1$.

(c) Let

$$g_h(x) = \frac{hf(x)}{f(x+h) - f(x)} = \frac{x(x-1)^2}{3x^2 + (3h-4)x + (h-1)^2}$$

By setting

$$\phi_h(x) = 3x^2 + (3h-4)x + (h-1)^2$$

and direct computation, we obtain

$$g_h'(0) = \frac{f'(0)\phi_h(0) - f(0)\phi_h'(0)}{\phi_h^2(0)} = \frac{1}{(h-1)^2}$$

Once

$$|1 - g_h'(0)| < 1,$$

the iteration converges. Therefore, the feasible set for h is as follows:

$$\left\{h : h > \frac{\sqrt{2}}{2} + 1 \ or \ 0 < h < -\frac{\sqrt{2}}{2} + 1\right\}.$$

The convergence order is 1 since $\phi'(0) \neq 0$.

(d) The solution is not unique. What follows is just an example: Let

$$\varphi(x) = x - \gamma\frac{f(x)}{f'(x)}$$

with $\gamma \in (0, 2)$. As

$$|\varphi'(1)| = |1 - \gamma| < 1,$$

the iteration converges.

□