MATH3230A - Numerical Analysis
Exercises on Polynomial Interpolation

1. Find the polynomials of least degree that interpolate these set of points:
   (a) $x_0 = 3,\ f(x_0) = 5,\ x_1 = 7,\ f(x_1) = -1$;
   (b) $x_0 = 3,\ f(x_0) = 12,\ x_1 = 7,\ f(x_1) = 146,\ x_2 = 1,\ f(x_2) = 2$;
   (c) $x_0 = 1.5,\ f(x_0) = 0,\ x_1 = 2.7,\ f(x_1) = 0,\ x_2 = 3.1,\ f(x_2) = 0,\ x_3 = -2.1,\ f(x_3) = 1$;
2. Show that if $p(x)$ is a polynomial of degree $\leq n$ interpolating the function $f(x)$ at distinct points $x_0 < x_1 < \cdots < x_n$, then
$$f(x) - p(x) = \sum_{i=0}^{n} [f(x) - f(x_i)] l_i(x).$$
3. Show that if g interpolates f at points $x_0, \ldots, x_{n-1}$ and h interpolates f at $x_1, \ldots, x_n$, then
$$g(x) + \frac{x_0 - x}{x_n - x_0}(g(x) - h(x))$$
   interpolates f at $x_0, \ldots, x_n$.
4. Show that interpolating $f(x) = \cosh(x)$ with a polynomial $p(x)$ of degree 22 with 23 distinct points in $[-1, 1]$ gives a relative error no greater than 5×10^{-16}.
5. Show that for the Chebyshev polynomial $T_k(x)$ for $x \geq 1$ can be expressed as
$$T_k(x) = \cosh(k \cosh^{-1}(x)).$$
6. Find the best upper bound on the error of interpolation $f(x) = e^{x-1}$ with a polynomial of degree 12 using 13 nodes in $[-1, 1]$.
7. Compute the table of divided differences for the following data set:
   (a) $x \in \{0, 1, 2, 3, 4\},\ f(x) \in \{2, 1, 2, -7, 10\}$;
   (b) $x \in \{4, 2, 0, 3\},\ f(x) \in \{64, 11, 7, 28\}$;
8. Show that if f is a polynomial of degree k, then for $n > k$,
$$f[x_0, \ldots, x_n] = 0.$$

9. Prove the Leibniz formula

$$(fg)[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f[x_0, \ldots, x_k]g[x_k, \ldots, x_n].$$

10. Find Hermite's interpolation for the data

$$p(x_i) = y_i, \quad p'(x_i) = 0 \text{ for } i = 0, 1, \ldots, n.$$

11. In the derivation of the Lagrange form of Hermite's interpolation, compute $l_i'(x)$ for $0 \le i \le n$.