MATH3230A - Numerical Analysis
Exercises on Linear systems of algebraic equations

1 Vector and matrix norms

1. Show that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ for all $x \in \mathbb{R}^n$.
2. Show that $\|x\|_1 \leq n\|x\|_\infty$ and $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$ for all $x \in \mathbb{R}$.
3. Consider for matrix $A = (a_{ij})_{1\leq i,j\leq n}$,

$$\|A\| = \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|.$$

   Is this a matrix norm?

4. Suppose for a matrix $A \in \mathbb{R}^{n\times n}$, $\|A\| < 1$. Show that there cannot exist $x \in \mathbb{R}^n$ with $\|x\| = 1$ such that $(I - A)x = 0$. Deduce that $(I - A)$ is invertible.
5. Compute

$$(I - A)\sum_{i=0}^{m} A^i,$$

   and show that if $\|A\| < 1$ then $\|A^{m+1}\| \leq \|A\|^{m+1} \to 0$. Deduce that $(I - A)^{-1} = \sum_{i=1}^{\infty} A^i$.

6. Show that if A is invertible, then for any B it holds that

$$\frac{\|I - AB\|}{\|A\|} \leq \|B - A^{-1}\|.$$

2 Condition number

1. For two invertible matrices A and B, show that

$$\kappa(AB) \leq \kappa(A)\kappa(B).$$

2. Compute the condition number of the following matrices using $\|\cdot\|_1$ and $\|\cdot\|_\infty$ norms:

$$(i)\begin{pmatrix} a+1 & a \\ a & a-1 \end{pmatrix},\quad (ii)\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix},\quad (iii)\begin{pmatrix} 4 & -3 \\ -1 & 0 \end{pmatrix}$$

3. Show that the condition number $\kappa(A)$ can also be expressed as

$$\kappa(A) = \sup_{\|x\|=\|y\|} \frac{\|Ax\|}{\|Ay\|}.$$

4. Show that the condition number satisfies the property

$$\kappa(\lambda A) = \kappa(A) \text{ for all } \lambda \neq 0.$$

3 Cholesky and LU factorizations

1. Factorize the following matrices

$$\begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 6 & 7 \end{pmatrix}$$

2. For the SPD matrix

$$\begin{pmatrix} 2 & 6 & -4 \\ 6 & 17 & -17 \\ -4 & -17 & -20 \end{pmatrix},$$

determine directly the LDU factorization from its Cholesky factorization..

3. Suppose a SPD matrix A has a Cholesky factorization $A = U^T U$. What can be said about its determinant?

4. If A is positive definite, does it follows that A^{-1} is also positive definite?

5. Define a P-matrix to be one where $a_{ij} = 0$ if $j \leq n - i$ and a Q-matrix to be a P-matrix in which $a_{i,n-i+1} = 1$ for $i = 1, 2, \ldots n$. Find a PQ factorization of the matrix

$$A = \begin{pmatrix} 3 & 15 \\ -1 & -1 \end{pmatrix}.$$

6. Given a LU factorization of the matrix A. Write down an algorithm to solve $A^T x = b$.

4 Pivoting

1. Use Gaussian elimination with partial pivoting to find the determinant of the matrix

$$\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

2. Let P be a permutation matrix, i.e., it is obtained by swapping certain rows of the identity matrix $I \in \mathbb{R}^{n\times n}$. Show that if L is a unit lower triangular matrix, then PLP^{-1} is also unit lower triangular.

3. Find a permutation matrix P, unit lower triangular matrix L, upper triangular matrix U so that $PA = LU$ for the following matrix

$$A = \begin{pmatrix} -1 & 6 & 0 & -8 \\ 1 & 0 & -6 & 1 \\ 0 & 1 & -1 & -1 \\ 2 & 0 & -4 & 4 \end{pmatrix}.$$

4. Use Gaussian elimination with pivoting to obtain the factorization $PA = LDU$ for the matrix

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 6 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$

5. A matrix $A \in \mathbb{R}^{n\times n}$ is column-wise diagonally dominant if

$$|a_{jj}| > \sum_{1\le i\le n,\ i\ne j} |a_{ij}| \text{ for all } 1 \le j \le n.$$

Let $A^{(1)}$ denote the matrix obtained from A after applying one round of Gaussian elimination with entries $a_{ij}^{(1)} = a_{ij} - \frac{a_{1j}}{a_{11}} a_{i1}$ and $a_{i1}^{(1)} = 0$ for $2 \le i \le n$. Show the following assertions to establish that $A^{(1)}$ is also column-wise diagonally dominant, i.e.,

$$|a_{jj}^{(1)}| > \sum_{i\ge 2, i\ne j} |a_{ij}^{(1)}| \text{ for } 2 \le j \le n.$$

(i) $\displaystyle\sum_{i\ge 2, i\ne j} |a_{ij}^{(1)}| \le \sum_{i\ge 2, i\ne j} |a_{ij}| + \frac{|a_{1j}|}{|a_{11}|} \sum_{i\ge 2, i\ne j} |a_{i1}|$;

(ii) $\displaystyle\sum_{i\ge 2, i\ne j}^{n} |a_{ij}| \le |a_{jj}| - |a_{1j}|$;

(iii) $\displaystyle\sum_{i\ge 2, i\ne j}^{n} |a_{i1}| \le |a_{11}| - |a_{j1}|$;

(iv) $\displaystyle |a_{jj}| - \frac{|a_{1j}|}{|a_{11}|} |a_{j1}| \le |a_{jj}^{(1)}|$.

Is partial pivoting needed for matrices that are column-wise diagonally dominant?

5 Non-square linear systems

1. The equation of a straight line is given by $y = mx + c$. Find the closest line to the points $(0, 6)$, $(1, 0)$ and $(2, 0)$ by defining a suitable overdetermined system. [Here closest means the 2-norm of the error is minimized]

2. Find a parabola given by $y = a + bx + cx^2$ that comes closest to the values $y = (0, 0, 1, 0, 0)$ at points $x = (-2, -1, 0, 1, 2)$ by defining a suitable overdetermined system.

3. Compute the point (x, y, x) that lies on the intersection of two planes

$$x + y + x = 1, \quad -x - y + x = 0$$

   that is closest to the origin by defining a suitable underdetermined system.

4. Show that the following systems has infinitely many solutions:

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 2, \\ 2x_1 + x_2 - 2x_3 &= -1, \\ 3x_1 - x_2 + 2x_3 &= 1, \\ 2x_1 + 6x_2 - 12x_3 &= -6. \end{aligned}$$

   Then, compute the solution with the smallest 2-norm.