MATH3230A - Numerical Analysis
Exercises on Nonlinear equations in one variable

1 Rate of convergence

For the following sequences compute their limits (if they exist) and state the type of convergence.

1. $x_k = 1 + 2^{2-k}$ for $k = 0, 1, 2, \ldots$;
2. $x_{k+1} = 2^{-k} x_k$ for $k = 0, 1, 2, \ldots$;
3. $x_k = k^{-d}$ for fixed $d > 0$;
4. $x_{k+1} = 10^{-4^k}$ for $k = 0, 1, 2, \ldots$
5. $x_k = (0.5)^{3^k}$ for $k = 0, 1, 2, \ldots$;
6. $x_k = k^{1/k}$ for $k = 1, 2, \ldots$;
7. $x_{k+1} = \tan^{-1}(x_k)$;

2 Bisection algorithm

1. Apply the Bisection algorithm to find a positive root of

   $$x^2 - 4x\sin(x) + (2\sin(x))^2 = 0.$$

   Use a calculator to determine the root up to 2 significant decimal points.

2. Let $[a_0, b_0]$ denote the initial interval, and let $[a_k, b_k]$ denote the interval at the kth iteration of the Bisection algorithm. Give an example of a function $f(x)$ such that

   $$a_0 = a_1 < a_2 = a_3 < a_4 = a_5 < a_6 = \cdots.$$

3. For $k \in \{2, 3, 4\}$ consider the function $f_k(x) = (x-1)^k$. Explain why the Bisection method does not work for the case $k = 2$ and $k = 4$.

4. For a non-negative differentiable function $f : \mathbb{R} \to \mathbb{R}$ with a single minimum at x^*, i.e., $f(x^*) \leq f(x)$ for all $x \in \mathbb{R}$, what conditions are needed for the function f in order to apply the Bisection algorithm to find the value of x^*?

3 Newton's method

1. Compute the first three Newton iterates for the nonlinear problem

$$f(x) = 4x^3 - 2x^2 + 3 = 0$$

with $x_0 = -1$.

2. Define $x_0 = 0$ and

$$x_{k+1} = x_k - \frac{\tan(x_k) - 1)}{\sec^2(x_k)}.$$

What is the limit of the sequence $(x_k)_{k\in\mathbb{N}}$ and what is the relation to Newton's method?

3. Devise a Newton iteration formula for computing $R^{1/5}$ for a given value $R > 0$.

4. Explain why Newton's method will not converge for the nonlinear problem $f(x) = x^2 + 1 = 0$ for any starting value $x_0 \in \mathbb{R}$.

5. Halley's method for solving $f(x) = 0$ uses the formula

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{(f'(x_k))^2 - (f(x_k)f''(x_k))/2}.$$

Show that this is Newton's method applied to the function $f(x)/\sqrt{f'(x)}$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing convex function with $f(x^*) = 0$ for some x^*. Here convex means that $f''(x) > 0$ for all $x \in \mathbb{R}$. Show that for any starting point $x_0 \in \mathbb{R}$ the Newton's method applied to the problem $f(x) = 0$ always converge to x^* by establishing the following properties

   - $e_{k+1} < e_k$ for all k, where $e_k = x_k - x^*$;
   - $e_k > 0$ for all k;
   - the sequences $(e_k)_{k\in\mathbb{N}}$ and $(x_k)_{k\in\mathbb{N}}$ converge and identify the equations the limits fulfill.

7. Steffensen's method considers the iteration formula

$$x_{k+1} = x_k - \frac{f(x_k)}{g(x_k)}, \quad g(x) := \frac{f(x + f(x)) - f(x)}{f(x)}.$$

Show that this converges quadratically under suitable hypothesis.

4 Fixed-point iterative methods

1. What is x_2 if $x_0 = 1$, $x_1 = 2$, $f(x_0) = 2$ and $f(x_1) = 1.5$ in the application of the Secant method?

2. What is x_2 if $x_0 = 0$, $x_1 = 1$ in the application of the Secant method to $f(x) = x^2 - 2$?

3. Write out the Secant method applied to the problem $f(x) = e^x - \tan(x) = 0$.

4. Compute x_2, x_3, x_4 for the Secant method applied to $f(x) = x^3 - \sinh(x) + 4x^2 + 6x + 9$ with $x_0 = 8$, $x_1 = 7$.

5. For an iterative function of the form $\varphi(x) = x + f(x)g(x)$ with $f(x^*) = 0$ and $f'(x^*) \neq 0$. Find sufficient conditions for g so that the fixed-point iteration method will converge cubically to x^* if if started close to x^*.

6. Let $I \subset \mathbb{R}$ be an interval. Recall that a mapping $F : I \to \mathbb{R}$ is contractive if there is a constant $\lambda \in (0,1)$ such that

$$|F(x) - F(y)| \leq \lambda |x - y|$$

for all $x, y \in I$. Show that the following functions are contractive, and give an estimate on the value of λ:

- $F(x) = x/2$ for $I = [1,5]$;
- $F(x) = |x|^{3/2}$ for $I = [-1/3, 1/3]$;
- $F(x) = (1 + x^2)^{-1}$ for any interval I;

7. The Contraction Mapping Theorem states that if $F : I \to I$ is a contractive mapping where I is a closed subset of $\mathbb{R}$, then F has a unique fixed point. Show that (i) the function $F(x) = 4x(1-x)$ maps the interval $I = [0,1]$ to itself, (ii) F is not a contraction. Explain why this does not contradict with the Contraction Mapping Theorem.

8. Let p be a positive number. Define a fixed-point iterative method to compute the value

$$x = \sqrt{p + \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}}.$$

9. Express the Steffensen method as a fixed-point iterative method and compute its order of convergence.