1. Given $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, solve the determinant equation $\begin{vmatrix} \lambda-0 & -1 \\ -1 & \lambda-0 \end{vmatrix} = 0$ for λ.

Sol: $\begin{vmatrix} \lambda-0 & -1 \\ -1 & \lambda-0 \end{vmatrix} = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = 0$

$\Rightarrow \lambda = \pm 1$

Remark: $Ax = \lambda x$, $x \neq 0$ (definition of eigenvalues)

$\Leftrightarrow Ax = \lambda I x$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, (identity matrix)

$\Leftrightarrow Ax - \lambda I x = 0$

$\Leftrightarrow (A - \lambda I)x = 0$, $x \neq 0$ (distribution law of matrix arithmetics)

$\Leftrightarrow A - \lambda I$ is singular

$\Leftrightarrow \det(A - \lambda I) = 0$

This means solving $|A - \lambda I| = 0$ (or $|\lambda I - A| = 0$), is indeed finding the eigenvalues of A.

2. Find the eigenvectors of A.

Sol: • First notice that different eigenvalues should have different eigenvectors.

Otherwise, say $\lambda_1 = 1$, $\lambda_2 = -1$ have the same eigenvector x,

then $\begin{cases} Ax = \lambda_1 x \\ Ax = \lambda_2 x, \ x \neq 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 x = \lambda_2 x \\ x \neq 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2$. (Contradiction).

• Second, eigenvectors are not unique.

Say, $Ax = \lambda x$, then $A(2x) = 2Ax = 2(\lambda x) = \lambda \cdot (2x)$.

$\Rightarrow$ $2x$ is also an eigenvector !

• Third, computation issue :

For $\lambda = 1$, $Ax = \lambda x$

$\Leftrightarrow (A - \lambda I)x = 0$

$\Leftrightarrow \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. (write $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$)

$$\Leftrightarrow \begin{cases} -\lambda x_1 + x_2 = 0 \\ x_1 - \lambda x_2 = 0 \end{cases} \quad \text{(matrix multiplication)}$$

$$\Leftrightarrow \begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases} \quad \text{(put } \lambda = 1 \text{ inside)}$$

$$\Leftrightarrow \quad x_1 = x_2 .$$

So any vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ of the form $x_1 = x_2$, is an eigenvector of A, which corresponds to $\lambda = 1$.

i.e. $x = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $x_1 \in \mathbb{R} \setminus \{0\}$, are desired eigenvectors.

Simply pick $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(Verify it : $Ax = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \cdot x$)

For $\lambda = -1$: $\quad Ax = \lambda x \Leftrightarrow \begin{cases} -\lambda x_1 + x_2 = 0 \\ x_1 - \lambda x_2 = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases} \Leftrightarrow \quad x_1 = -x_2 .$$

So $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $x_2 \in \mathbb{R}$.

Pick $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(Verify it : $Ax = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda \cdot x$)

3. Repeat 1, 2 for $B = \begin{pmatrix} -5 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & -2 & -5 \end{pmatrix}$

Sol : $\det(B - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} -5-\lambda & 2 & -1 \\ 2 & -2-\lambda & -2 \\ -1 & -2 & -5-\lambda \end{vmatrix} = 0$

$$\Leftrightarrow (-5-\lambda)\begin{vmatrix} -2-\lambda & -2 \\ -2 & -5-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & -2 \\ -1 & -5-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & -2-\lambda \\ -1 & -2 \end{vmatrix} = 0$$

$$\Leftrightarrow (-5-\lambda)(\lambda^2+7\lambda+10-4) - 2(-10-2\lambda-2) - (-4-2-\lambda) = 0$$

$$\Leftrightarrow -(\lambda+5)(\lambda^2+7\lambda+6) + 4(\lambda+6) + (\lambda+6) = 0$$

$$\Leftrightarrow -(\lambda+5)(\lambda+1)(\lambda+6) + 4(\lambda+6) + (\lambda+6) = 0$$

$$\Leftrightarrow (\lambda+6)\left[-\lambda^2-6\lambda-5+4+1\right] = 0$$

$$\Leftrightarrow \quad (\lambda+6)\cdot\left[-\lambda^2-6\lambda\right]=0$$

$$\Leftrightarrow \quad \lambda(\lambda+6)^2=0$$

$$\Rightarrow \quad \lambda=0 \quad \text{or} \quad \lambda=-6$$

For $\lambda=0$: $\quad (B-\lambda I)\vec{x}=\vec{0}$

$$\Leftrightarrow \begin{pmatrix}-5 & 2 & -1\\ 2 & -2 & -2\\ -1 & -2 & -5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}-5 & 2 & -1\\ 0 & -\frac{6}{5} & -\frac{12}{5}\\ 0 & -\frac{12}{5} & -\frac{24}{5}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}-5 & 2 & -1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

(This is a process called Gaussian elimination. A method to solve $Ax=b$)

$$\Leftrightarrow \begin{pmatrix}-5 & 0 & -5\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow \begin{cases}x_1=-x_3\\ x_2=-2x_3\end{cases}$$

So $\vec{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-x_3\\-2x_3\\x_3\end{pmatrix}=x_3\cdot\begin{pmatrix}-1\\-2\\1\end{pmatrix}$. $\quad$ Pick $\vec{x}=\begin{pmatrix}-1\\-2\\1\end{pmatrix}$.

For $\lambda=-6$. $\quad (B-\lambda I)\vec{x}=\vec{0}$

$$\Leftrightarrow \begin{pmatrix}1 & 2 & -1\\ 2 & 4 & -2\\ -1 & -2 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}1 & 2 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Leftrightarrow \quad x_1+2x_2-x_3=0 \quad \Leftrightarrow \quad x_3=x_1+2x_2.$$

$$\Rightarrow \quad \vec{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\x_1+2x_2\end{pmatrix}=\begin{pmatrix}x_1\\0\\x_1\end{pmatrix}+\begin{pmatrix}0\\x_2\\2x_2\end{pmatrix}$$

$$=x_1\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}+x_2\cdot\begin{pmatrix}0\\1\\2\end{pmatrix}. \qquad x_1, x_2\in\mathbb{R}.$$

Pick $\vec{x}=\begin{pmatrix}1\\0\\1\end{pmatrix}$ or $\vec{x}=\begin{pmatrix}0\\1\\2\end{pmatrix}$, they are both eigenvectors of A corresponding to $\lambda=-6$.