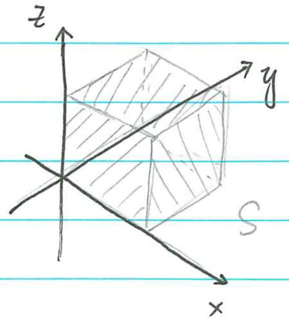
1. $\vec{F} = (xz,\ yz,\ x^2+y^2)$

$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(x^2+y^2) = z+z+0 = 2z$

• Flux of $\vec{F}$ out of the surface $= \int_S \vec{F}\cdot\vec{n}\, dS$



By Divergence Thm,

$$\int_S \vec{F}\cdot\vec{n}\, dS = \int_V \nabla\cdot\vec{F}\, dV$$

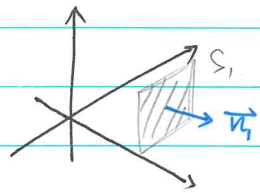
where V is the region enclosed by S.

$$\int_V \nabla\cdot\vec{F}\, dV = \int_0^1\int_0^1\int_0^1 2z\, dx\, dy\, dz = 1$$

Remark: You can do 6 surface integral to get the result.

Take $S_1 = \{(x,y,z)\in\mathbb{R}^3 : x=1,\ 0\le y\le 1,\ 0\le z\le 1\}$ as an example.

$\vec{n}_1 = (1,0,0)$ is the unit outward normal on S_1.



$$\int_{S_1} \vec{F}\cdot\vec{n}_1\, dS_1 = \int_0^1\int_0^1 (xz, yz, x^2+y^2)\cdot(1,0,0)\, dy\, dz$$
$$= \int_0^1\int_0^1 xz\, dy\, dz = \int_0^1\int_0^1 z\, dy\, dz$$
$$= \frac{1}{2}$$

2. $\vec{F} = (x^3, -y, z)$

Similar to Q1,

$$\text{Flux} = \int_S \vec{F}\cdot\vec{n}\, dS = \int_V \nabla\cdot\vec{F}\, dV$$

$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z) = 3x^2 - 1 + 1 = 3x^2$

$$\int_V \nabla\cdot\vec{F}\, dV = \int_0^{2\pi}\int_0^1 3x^2\cdot r\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^1 3r^2\cos^2\theta\cdot r\, dr\, d\theta$$
$$= \left(\int_0^{2\pi}\cos^2\theta\, d\theta\right)\cdot\left(\int_0^1 3r^3\, dr\right)$$
$$= \int_0^{2\pi}\frac{\cos 2\theta + 1}{2}\, d\theta\cdot\frac{3}{4} = \frac{3\pi}{4}$$

<u>Remarks</u> : (1). You can still use x, y, z coordinates for calculation.

$$\int_V \nabla\cdot\vec{F}\,dV = \int_0^1\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3x^2\,dy\,dx\,dz$$

$$= \int_0^1\int_{-1}^1 3x^2\cdot 2\sqrt{1-x^2}\,dx\,dz = \int_{-1}^1 6x^2\sqrt{1-x^2}\,dx$$

(Need to use trigonometric substitution, be careful about the domain)

(2). You can do 3 surface integrals for calculation as well.

For  , one needs to find out $\vec{n}$ and dS

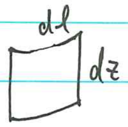
① Geometric method :

- $\vec{n}$: Observe $\vec{n}$ must be parallel to ~~xz~~ xy-plane, $\Rightarrow n_3 = 0$ $\quad (\vec{n} = (n_1, n_2, n_3))$

and it must be perpendicular to the unit circle $(\cos\theta, \sin\theta, z) = X(\theta, z)$.

$$\Rightarrow \quad \vec{n} = (\cos\theta, \sin\theta, 0)$$

<u>Note</u> : $\vec{n}\cdot\frac{\partial X}{\partial\theta} = 0$

- ds : 

Area element $= dl\cdot dz$.

Note : $\frac{dl}{r} = d\theta$ (def. of radian)

$$\Rightarrow \quad dS = r\,d\theta\,dz = d\theta\,dz \qquad (r=1)$$

② Use charts. (More advanced)

$X(\theta, z) = (\cos\theta, \sin\theta, z)$

$$\left.\begin{array}{l} X_\theta = (-\sin\theta, \cos\theta, 0) \\ X_z = (0, 0, 1) \end{array}\right\} \text{tangent vectors.}$$

$$\vec{n} = X_\theta\times X_z = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \sin\theta, 0)$$

$$g = \begin{pmatrix} X_\theta\cdot X_\theta & X_\theta\cdot X_z \\ X_z\cdot X_\theta & X_z\cdot X_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$dS = \sqrt{\det g}\,d\theta\,dz = d\theta\,dz.$$

3. $\vec{F} = (z, x, y)$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y) = 0$$

$$\text{Flux} = \int_S \vec{F} \cdot \vec{n} \, dS = \int_V \nabla \cdot \vec{F} \, dV = 0$$