MATH2550

Course Work

1. Compute $\int_C x\, ds$, where C is the part of the circle $x^2 + y^2 = a^2$ running from $(0, a)$ to $(a, 0)$ and finally to $\left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$.

2. Using Green's Theorem, compute $\int_C 3x^2 y dx - x^3 dy$, where C is the piecewise-straight-line curve formed by joining $(0,0)$ to $(1,0)$ and finally to $(1,1)$.

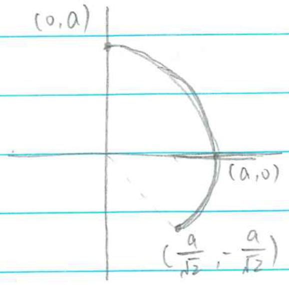
3. The following lines are typical errors students made when using Green's Theorem. Explain what the mistake(s) is/are in each case (here C is the circle of radius 1 running counter-clockwise):

   a) $\int_C y^2 x dx - x^2 y dx = \iint_{x^2+y^2\leq 1} [y^2 - (-x^2)] dxdy$

   b) $\int_C y^2 x dx - x^2 y dx = -\iint_{x^2+y^2\leq 1} [-2xy - 2xy] dxdy$

4. Compute $\int_C (12xy + e^y) dx - (\cos y - xe^y) dy$, where C is the curve running first from $(-1,1)$ to $(0,0)$ joined by the parabola $y = x^2$, followed by the line segment joining $(0,0)$ to $(2,0)$.
   (You may find the following "trick" useful – join the two endpoints of the curve, i.e. $(-1,1)$ and $(2,0)$ by some line segments and form a closed curve C^*. Then use Green's Theorem)

1.



① Define $\alpha : [-\frac{\pi}{4}, \frac{\pi}{2}] \longrightarrow \mathbb{R}^2$ by

$\alpha(t) = (a\cos t,\ a\sin t)$

② $\left|\frac{d\alpha}{dt}\right| = |(-a\sin t,\ a\cos t)|$

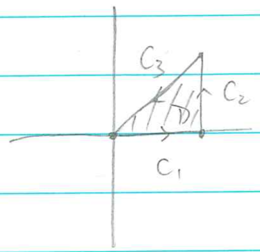
$= \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$

③ $\int_C x\, ds = \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} a\cos t \cdot a\, dt \quad \left(= \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} x(t)\,|\alpha'(t)|\, dt\right)$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} a^2 \cos t\, dt$

$= a^2 \sin t \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{2}} = a^2\left(1 - \left(-\frac{\sqrt{2}}{2}\right)\right) = \frac{2+\sqrt{2}}{2} a^2$

2.



$C = C_1 + C_2$

let $M = 3x^2y$, $N = -x^3$

$\int_C M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$

$= \int_{C_1+C_2+C_3} M\,dx + N\,dy - \int_{C_3} M\,dx + N\,dy$

By Green's thm,

$\int_{C_1+C_2+C_3} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \iint_D (-3x^2 - 3x^2)\, dx\,dy$

$= \iint_D -6x^2\, dx\,dy$

Parametrize D : $0 \le x \le 1$, $0 \le y \le x$

$\iint_D -6x^2\, dx\,dy = \int_0^1 \int_0^x -6x^2\, dy\, dx$

$= \int_0^1 -6x^3\, dx = -\frac{6x^4}{4}\Big|_0^1 = -\frac{3}{2}$

Parametrize C_3 : $\alpha(t) = (1,1) + t(-1,-1)$, $t \in [0,1]$

$= (1-t,\ 1-t)$

$\alpha'(t) = (-1,-1)$ ~~$\Rightarrow |\alpha'(t)| = \sqrt{1^2+1^2} = \sqrt{2}$~~

$\int_{C_3} M\,dx + N\,dy = \int_{C_3} \vec{F}\cdot d\vec{r}$, where $\vec{F} = (M, N)$

$= \int_0^1 \underbrace{(3x(t)^2 \cdot y(t),\ -3x(t)^3)}_{\vec{F}(t)} \cdot \underbrace{(-1,-1)}_{\alpha'(t)}\, dt$

~~$= \int_0^1 (3(1-t)^3,\ -3(1-t)^3)\cdot(-1,-1)$~~

$= \int_0^1 (3(1-t)^3, -3(1-t)^3) \cdot (-1,-1)\, dt$

$= \int_0^1 3(1-t)^3\cdot(-1) + (-3(1-t)^3)\cdot(-1)\, dt = 0$

So $\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy - \int_{C_3} M\,dx + N\,dy$
$= -\frac{3}{2} - 0 = -\frac{3}{2}$

Remarks:

① $\int_C \vec{F} \cdot d\vec{r} \overset{def}{=} \int_C M\,dx + N\,dy = \int_{t_0}^{t_1} (M(t), N(t)) \cdot (x'(t), y'(t))\,dt$

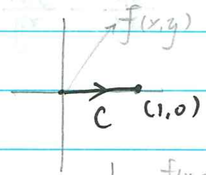
where $\alpha(t) = (x(t), y(t))$, $t_0 \le t \le t_1$ is a parametrization of C.

② For scalar-valued f, orientation of C does NOT matter when calculating $\int_C f\,ds$

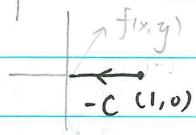
For vector-valued f, orientation of C is IMPORTANT:

$\int_C f \cdot d\vec{r} = -\int_{-C} f \cdot d\vec{r}$, where $-C$ denotes the "reverse" of C

Example: $C: \alpha(t) = (t, 0)$, $0 \le t \le 1$,



$-C: \beta(t) = (-t+1, 0)$, $0 \le t \le 1$



$f(x,y) = (1,1)$

$\int_C f \cdot d\vec{r} = \int_0^1 (1,1) \cdot \alpha'(t)\,dt = \int_0^1 (1,1)\cdot(1,0)\,dt$
$= \int_0^1 1 + 0\,dt = 1$

$\int_{-C} f \cdot d\vec{r} = \int_0^1 (1,1) \cdot \beta'(t)\,dt = \int_0^1 (1,1) \cdot (-1,0)\,dt$
$= \int_0^1 -1 + 0\,dt = -1$

Note: This is consistent with the work done by the force f!
Along C, the work done by f should be > 0;
Along $-C$, the work done by f should be < 0.

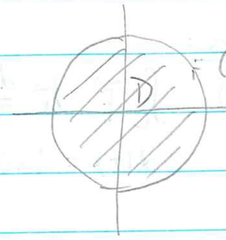
③ You need to have a region to apply Green's thm. (Like Q2)

3. (a) Formula : $\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

It does not make sense to have $\int_C M\,dx + N\,dx$ as C is not a straight line on x-axis.

So $\int_C y^2x\,dx - x^2y\,dy = \iint_D -2xy - 2yx\,dx\,dy$

$= \iint_D -4xy\,dx\,dy$



or : $\int_C \underbrace{y^2x}_{N}\,dy - \underbrace{x^2y}_{M}\,dx = \iint_D y^2 - (-x^2)\,dx\,dy$

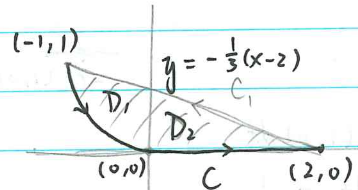
$= \iint_D x^2 + y^2\,dx\,dy$

(b) Similar ~~as~~ to (a).

Remark: A way to remember $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ here :

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ M & N \end{vmatrix}$ (operator on the 1st row)

4.



Let C_1 be the straight line (segment) joining (2,0) to (-1,1).

$\alpha(t) = (2,0) + t(-3,1), \quad 0 \le t \le 1$

$= (2-3t,\ t)$

$\int_C \underbrace{(12xy + e^y)}_{M}\,dx - \underbrace{(\cos y - xe^y)}_{N}\,dy = \int_{C+C_1} M\,dx + N\,dy - \int_{C_1} N\,dy + M\,dx$

$= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy - \int_{C_1} M\,dx + N\,dy$

$\iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy = \iint_D e^y - (12x + e^y)\,dx\,dy = \iint_D -12x\,dx\,dy$

Decompose D into D_1, D_2.

D_1 : $-1 \le x \le 0, \quad x^2 \le y \le -\frac{1}{3}(x-2)$

D_2 : $0 \le x \le 2, \quad 0 \le y \le -\frac{1}{3}(x-2)$

頁次
Page . . . .

$$\iint_{D_1} -12x\,dx\,dy = \int_{-1}^{0}\int_{x^2}^{-\frac{1}{3}(x-2)} -12x\,dy\,dx$$
$$= \int_{-1}^{0} -12x\left(-\tfrac{1}{3}(x-2) - x^2\right)dx = \int_{-1}^{0} 12x^3 + 4x^2 - 8x\,dx$$
$$= \left(3x^4 + \tfrac{4}{3}x^3 - 4x^2\right)\Big|_{-1}^{0}$$
$$= -3 + \tfrac{4}{3} + 4 = \tfrac{7}{3}$$

$$\iint_{D_2} -12x\,dx\,dy = \int_0^2\int_0^{-\frac{1}{3}(x-2)} -12x\,dy\,dx$$
$$= \int_0^2 -12x\cdot\left(-\tfrac{1}{3}(x-2)\right)dx = \int_0^2 4x^2 - 8x\,dx$$
$$= \left(\tfrac{4}{3}x^3 - 4x^2\right)\Big|_0^2$$
$$= \tfrac{32}{3} - 16 = -\tfrac{16}{3}$$
$$\iint_D -12x\,dx\,dy = \iint_{D_1} -12x\,dx\,dy + \iint_{D_2} -12x\,dx\,dy = \tfrac{7}{3} - \tfrac{16}{3} = -\tfrac{9}{3} = -3$$

$$\int_{C_1} M\,dx + N\,dy = \int_0^1 (M(t), N(t))\cdot \alpha'(t)\,dt$$
$$= \int_0^1 (M(t), N(t))\cdot(-3, 1)\,dt$$
$$= \int_0^1 -3\cdot\left(12x(t)y(t) + e^{y(t)}\right) + x(t)e^{y(t)} - \cos(y(t))\,dt$$
$$= \int_0^1 -3\left(12\cdot(2-3t)\cdot t + e^t\right) + (2-3t)e^t - \cos t\,dt$$
$$= \int_0^1 (-1-3t)e^t - \cos t + (108t - 72)t\,dt$$
$$\int_0^1 108t^2 - 72t\,dt = 36t^3 - 36t^2\Big|_0^1 = 0$$
$$\int_0^1 -\cos t\,dt = -\sin t\Big|_0^1 = -\sin 1$$
$$\int_0^1 -(3t+1)e^t\,dt = -(3t+1)e^t\Big|_0^1 + \int_0^1 e^t\,d(3t+1)$$
$$= -4e + 1 + 3\int_0^1 e^t\,dt = -4e + 1 + 3(e-1)$$
$$= -e - 2$$
$$\int_{C_1} M\,dx + N\,dy = -e - 2 - \sin 1$$

Hence $\displaystyle\int_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy - \int_{C_1} M\,dx + N\,dy$
$$= -3 + (e + 2 + \sin 1)$$
$$= e + \sin 1 - 1$$