1. Find the inverse of $A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$ by solving $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Sol: $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ \frac{a}{2} & \frac{b}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad a = d = 0, \quad b = 2, \quad c = 1$

2. Find the eigenvalues & eigenvectors of $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Sol: $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$\iff \lambda^2 - 4\lambda + 4 - 1 = 0$

$\iff \lambda^2 - 4\lambda + 3 = 0 \iff (\lambda - 1)(\lambda - 3) = 0$

So $\lambda = 1$ or $\lambda = 3$

- For $\lambda = 1$, $Ax = \lambda x$

$$\iff \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\iff \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{cases} (2-\lambda)x_1 - x_2 = 0 \\ -x_1 + (2-\lambda)x_2 = 0 \end{cases} \quad (*)$$

Substitute $\lambda = 1$ in $(*)$: $\begin{cases} x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{cases} \iff x_1 = x_2$

Take $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$

- For $\lambda = 3$, substitute $\lambda = 3$ in $(*)$:

$$\begin{cases} -x_1 - x_2 = 0 \\ -x_1 - x_2 = 0 \end{cases} \iff x_1 = -x_2$$

Take $x = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

3. What angles do the 2 eigenvectors of A make?

Sol: For $\vec{x_1} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{x_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

let θ be the angle between $\vec{x_1}$ and $\vec{x_2}$. $\theta \in [0, \pi]$

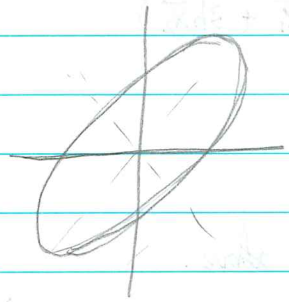
$$\cos\theta = \frac{\vec{x_1} \cdot \vec{x_2}}{\|\vec{x_1}\| \cdot \|\vec{x_2}\|}$$

Note: $\vec{x_1} \cdot \vec{x_2} = \frac{1}{2} \cdot (1 \cdot 1 + 1 \cdot (-1)) = 0$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} = 90^\circ$

4. Consider $(x \ y) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6$.

Draw this curve. What geometric object is this?



$2x^2 - 2xy + 2y^2 = 6$

$\Leftrightarrow (x - \frac{y}{2})^2 + \frac{3}{2} y^2 = 3$

Try different y find x.

An ellipse.

5. If we change coordinates for a simpler equation, what new axes would you choose?

Sol: ① From arithmetics: $(x \ y) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2x^2 - 2xy + 2y^2$

(Need to eliminate the cross term $-xy$.
The highest order is 2 for both x and y.)

Try $X = x + y$, $Y = x - y$.

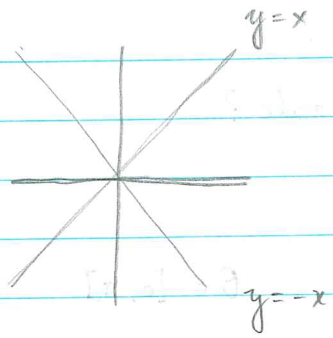
Aim to find $aX^2 + bY^2 = 2x^2 - 2xy + 2y^2$.

$\Rightarrow a \cdot (x^2 + 2xy + y^2) + b \cdot (x^2 - 2xy + y^2) = 2x^2 - 2xy + 2y^2$

$(a+b)(x^2 + y^2) + 2(a-b)xy = 2x^2 + 2y^2 - 2xy$.

Compare coefficient

$\Rightarrow \begin{cases} a + b = 2 \\ a - b = 1 \end{cases} \Rightarrow \begin{cases} a = \frac{3}{2} \\ b = \frac{1}{2} \end{cases}$



So the new coordinates are

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

② Use previous results:

By Q3, $\vec{x_1} \perp \vec{x_2}$, so for all $\vec{x} \in \mathbb{R}^2$,

$\vec{x} = a\vec{x_1} + b\vec{x_2}$, $a, b \in \mathbb{R}$ ($\vec{x_1}, \vec{x_2}$ span a plane)

Note $(x \;\; y) A \begin{pmatrix} x \\ y \end{pmatrix} = \vec{x}^T A \vec{x}$ ⊕

$$\begin{aligned}
&= \vec{x}^T A (a\vec{x_1} + b\vec{x_2}) \\
&= \vec{x}^T (a A\vec{x_1} + b A\vec{x_2}) = \vec{x}^T (a\vec{x_1} + 3b\vec{x_2}) \\
&= (a\vec{x_1}^T + b\vec{x_2}^T)(a\vec{x_1} + 3b\vec{x_2}) \\
&= a^2 \vec{x_1}\cdot\vec{x_1} + 4ab\, \vec{x_1}\cdot\vec{x_2} + 3b^2 \vec{x_2}^T\cdot\vec{x_2} \\
&= a^2 \vec{x_1}\cdot\vec{x_1} + 3b^2 \vec{x_2}^T\cdot\vec{x_2}
\end{aligned}$$

In terms of $\vec{x_1}, \vec{x_2}$, ⊕ is simplified as above.

So simply choose ~~x~~ $\vec{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{x_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as our new basis.