THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Tutorial 12
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Theorem 1 (Argument Principle)**.** *Let C be a closed contour and C encloses a domain U. Let f be meromorphic in U with poles p_i (i=1,...,n) and zeros z_j (j=1,...,m) counted according to multiplicity where p_i and z_j are inside U, then*

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}dz = \sum_{i=1}^{m} n(C, z_i) - \sum_{j=1}^{n} n(C, p_j)$$

Remark : $n(C, z_i)$ denotes the order of the poles or multiplicity of zeros inside the contour.

Remark : It can be viewed as $\frac{1}{2\pi i}\int_{z\in C}\frac{df}{f-0}$. It is the winding number of f around the point 0. It is similar to $\frac{1}{2\pi i}\int_{|z-a|=1}\frac{dz}{z-a} = 1$.

Theorem 2 (Rouché's Theorem (version 1))**.** *Let C be a closed contour and C encloses a domain U. Let f and g be analytic in U and on C. Suppose that $|f| > |g|$ on C, then f and $f+g$ have the same number of zeros inside C.*

Remark : Actually the Rouché's Theorem comes from the argument principle, but we do not require that the zeros of f and $f+g$ are inside the domain U as in argument principle. It is because the strict inequality on C exclude that the zeros are on C.

Remark : Theorem 2 and 3 only tell to you that the numbers of roots of f and $f+g$ are the same, but their roots may not be the same!

Remark : You should be careful that the inequality must be strict and it is enough to hold only on C.

Theorem 3 (Rouché's Theorem (more powerful version))**.** *Let C be a closed contour and C encloses a domain U. Let f and g be analytic in U and on C. Suppose that $|f| + |g| > |f+g|$ on C, then f and g have the same number of zeros inside C.*

Remark : Version 1 is powerful enough in many cases, but you can see that the second version is more powerful. We will mainly apply version 1.

Example 1. *Find the number of roots of the equation $2z^5 - 6z^2 + z + 1 = 0$ inside the circle $|z| = 2$.*

In the hypothesis of Rouché's theorem, we want to find a polynomial to dominate the other polynomial. Since the circle is $C = \{|z| = 2\}$, the term with largest power would dominate the other term. Hence we let $f = 2z^5$ and $g = -6z^2 + z + 1$. On C, we check that

$$|f| = 64 \text{ and } |g| \le 6(4) + 2 + 1 = 27.$$

Thus $|f| > |g|$ on C, f has 5 roots inside C. Therefore, $f + g = 2z^5 - 6z^2 + z + 1 = 0$ has 5 roots inside C.

Example 2. *Find the number of roots of the equation $2z^5 - 6z^2 + z + 1 = 0$ inside the circle $|z| = 1$.*

In this time, since the circle is $C = \{|z| = 1\}$, the term with largest coefficient would dominate the other term. Hence we let $f = -6z^2$ and $g = 2z^5 + z + 1$. On C, we check that

$$|f| = 6 \text{ and } |g| \le 2 + 1 + 1 = 4.$$

Thus $|f| > |g|$ on C, f has 2 roots inside C. Therefore, $f + g = 2z^5 - 6z^2 + z + 1 = 0$ has 2 roots inside C.

Example 3. *Find the number of roots of the equation* $f = z^4 + z^3 + 4z^2 + 2z + 3 = 0$ *in each quadrants.*

We would apply argument principle. Since polynomial with real coefficients have complex roots in conjugate pairs. Also the coefficients are positive, so the polynomial has no positive real roots.

We check that f has no purely imaginary roots. Let $z = yi$, then

$$f = y^4 - 4y^2 + 3 + i(2y - y^3).$$

If $f = 0$, we have $y = 0$ or $\pm\sqrt{2}$ from the imaginary part, but $f \neq 0$ by substituting these values of y into the real part of f.

We check that there is no negative real roots. On the real axis, we have

$$f' = 4z^3 + 3z^2 + 8z + 2 \text{ and } f'' = 12z^2 + 6z + 8 > 0.$$

From these, we see that f is convex and have local minima. Let x_1 such that $f'(x_1) = 4x_1^3 + 3x_1^2 + 8x_1 + 2 = 0$ and $f(x_1)$ is the global minimum. Then on the real axis,

$$\begin{aligned} f(z) \geq f(x_1) &= x_1^4 + x_1^3 + 4x_1^2 + 2x_1 + 3 \\ &= x_1^4 + \frac{1}{4}(4x_1^3 + 16x_1^2 + 8x_1 + 12) \\ &= x_1^4 + \frac{1}{4}(-(3x_1^2 + 8x_1 + 2) + 16x_1^2 + 8x_1 + 12) \\ &= x_1^4 + \frac{1}{4}(13x_1^2 + 10) > 0. \end{aligned}$$

Hence there is no negative real root.

It is enough to consider the number of roots in the right half plane since there is no root on the axis and the complex roots must be in conjugate pairs.

We take a contour C to be a right half circle with radius R,

$$\int_C \frac{f'}{f} dz = \int_{-\pi/2}^{\pi/2} + \int_{Ri}^{-Ri} \frac{f'}{f} dz.$$

For the first integral, $z = Re^{i\theta}$

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \frac{f'}{f} dz &= \int_{-\pi/2}^{\pi/2} \frac{4z^3 + 3z^2 + 8z + 2}{z^4 + z^3 + 4z^2 + 2z + 3} Re^{i\theta} i d\theta \\ &= i \int_{-\pi/2}^{\pi/2} \frac{4z^4 + 3z^3 + 8z^2 + 2z}{z^4 + z^3 + 4z^2 + 2z + 3} d\theta \\ &= i \int_{-\pi/2}^{\pi/2} \frac{4 + \frac{3}{z} + \frac{8}{z^2} + \frac{2}{z^3}}{1 + \frac{1}{z} + \frac{4}{z^2} + \frac{2}{z^3} + \frac{3}{z^4}} d\theta \\ &\to i \int_{-\pi/2}^{\pi/2} 4 d\theta = 4\pi i \quad \text{as} \quad R \to \infty \end{aligned}$$

For the second integral,

$$\int_{Ri}^{-Ri} \frac{f'}{f} dz = \int_{Ri}^{-Ri} \frac{df}{f - 0}.$$

To evaluate this integral, it is hard to compute it directly. However we know that this integral is the winding number of f around 0, that is, the times f winds the point 0 from $-Ri$ to Ri. If we set

$z = yi$, we have $f = y^4 - 4y^2 + 3 + i(2y - y^3)$. We find all the zero of the real and imaginary parts, that is $y = \pm 1, \pm\sqrt{3}$ and $y = 0, \pm\sqrt{2}$.

y	$-\infty$	$-\sqrt{3}$	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	$\sqrt{3}$	∞
Re(f)	+	0	-	0	+	0	-	0	+
Im(f)	0 *	+	0	-	0	+	0	-	0 *

Remark* : It is not really a zero, just for simplicity.

Thus, $\dfrac{1}{2\pi i}\displaystyle\int_{Ri}^{-Ri} \frac{f'}{f} dz \to -2$ as $R \to \infty$. Combining our results, $\displaystyle\int_C \frac{f'}{f} dz \to 0$ as $R \to \infty$, which implies that there is no root in the right half plane. Thus there are 0, 2, 2, 0 roots in quadrant I, II, III, IV respective.

Exercise: If $\lambda > 1$, find the number of roots of the equation $e^{-z} + z - \lambda = 0$ in the right half plane.

Good luck to your exam! ><