THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2018-19
Homework 7
Due Date: 1st November 2018

Compulsory Part

1. Find the units in the following rings:
   (a) $\mathbb{Z}$.
   (b) The ring R of all real valued functions on $\mathbb{R}$.
   (c) $R[x]$ where R is an integral domain.
2. Show that the set $R^\times$ of units in a ring R forms a group under multiplication.
3. Let R be an integral domain. Show that the polynomial ring $R[x]$ is also an integral domain.

Optional Part

1. (a) Is the product of two units in a ring necessarily a unit? If so, prove it; if not, provide a counterexample.
   (b) Is the sum of two units in a ring necessarily a unit? If so, prove it; if not, provide a counterexample.
2. Verify that under the convention $\deg 0 = -\infty$, the following rules hold for all polynomials $f, g \in R[x]$ where R is an integral domain:
   (a) $\deg(fg) = \deg f + \deg g$.
   (b) $\deg(f \pm g) \leq \max\{\deg f, \deg g\}$.
3. **Definition.** Let R be a ring. A subset S of R is said to be a **subring** of R if it is a ring under the addition $+$ and multiplication $\cdot$ associated with R, and its additive and multiplicative identity elements 0, 1 are those of R.

   To show that a subset S of a ring R is a subring, it suffices to show that:

   - $1_R \in S$,
   - $a - b \in S$ for any $a, b \in S$, and
   - S is closed under multiplication: $a \cdot b \in S$ for all $a, b \in S$.

   The **center** $Z(R)$ of R is defined as follows:

   $$Z(R) = \{r \in R : rs = sr \text{ for all } s \in R\}.$$

   Show that $Z(R)$ is a subring of R.

4. Let D be an integral domain. If there exists a positive integer n such that $na = \overbrace{a + \cdots + a}^{n \text{ times}} = 0$ for any $a \in D$, then D is said to be of **finite characteristic**; in this case, we define the **characteristic** of D to be

$$\text{char}(D) := \min\{n \in \mathbb{Z}_{>0} \mid na = 0 \forall a \in D\}.$$

If no such positive integer exists, we say that D is of **characteristic 0**, denoted as $\text{char}(D) = 0$.

   (a) Show that if $n1 \neq 0$ for any $n \in \mathbb{Z}_{>0}$, then D is of characteristic 0; otherwise, we have
   $$\text{char}(D) = \min\{n \in \mathbb{Z}_{>0} \mid n1 = 0 \forall a \in D\}.$$

   (b) Hence show that the characteristic of an integral domain is either 0 or a prime.