THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2018-19
Homework 1
Due Date: 13th September 2018

Compulsory Part

1. Let $\mathrm{SL}(2,\mathbb{Z})$ denote the set of 2×2 matrices with integer coefficients whose determinants are equal to 1. Prove that $\mathrm{SL}(2,\mathbb{Z})$ is a group under matrix multiplication.

2. Show that for all g, h in a group G, we have $(gh)^{-1} = h^{-1}g^{-1}$.

3. Let G_1, G_2 be groups. Show that the Cartesian product $G_1 \times G_2$ is a group under the operation
$$(a_1, b_1) * (a_2, b_2) := (a_1 *_1 a_2, b_1 *_2 b_2)$$
for $a_1, a_2 \in G_1$ and $b_1, b_2 \in G_2$, where $*_1, *_2$ are the group operations of G_1, G_2 respectively. The group $G_1 \times G_2$ is called the **direct product** of G_1 and G_2. Similarly, one can define the direct product of *any* number of groups.

Optional Part

1. Determine whether the given set equipped with the given binary operation is a group (if it is, give a proof; if it is not, explain why):

   (a) The set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers, equipped with addition.

   (b) The set $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$ of positive integers, equipped with multiplication.

   (c) The set $2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$ of even integers, equipped with addition. (How about odd integers?)

   (d) The set $U := \{z \in \mathbb{C} : |z| = 1\}$ of complex numbers with modulus 1, equipped with multiplication.

   (e) The set $V := \{z \in \mathbb{C} : |z| = 2\}$ of complex numbers with modulus 2, equipped with multiplication.

   (f) The set $M_{m\times n}(\mathbb{R})$ of $m \times n$ real matrices, equipped with matrix addition.

   (g) The set $M_{n\times n}(\mathbb{R})$ of real square matrices of size n, equipped with matrix multiplication.

   (h) The set $\{+, -\}$, equipped with the operation defined by $++ = +$, $+- = -$, $-+ = -$ and $-- = +$.

2. Let
$$S = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z} \right\}.$$
Is S a group under matrix multiplication? Justify your answer.

3. Let G be a group. Show that for any $a, b \in G$, the equation

$$ax = b$$

has a *unique* solution in G.

4. Let X be a nonempty set, and let S_X be the set of all bijective maps $\sigma : X \to X$. Show that S_X is a group under composition of maps. This is called the **symmetric group (or permutation group) on** X.

5. The **quaternion group** is defined as follows:

$$Q = \{1, -1, i, j, k, -i, -j, -k\},$$

where the group operation is written multiplicatively, the symbol 1 denotes the identity element, and $-i, -j, -k$ denotes $(-1)i$, $(-1)j$, $(-1)k$, respectively.

Moreover, by definition -1 commutes with every element of the group (for instance, $(-1)i = i(-1) = -i$), and the symbols i, j, k satisfy the following relations:

$$(-1)^2 = 1, \quad i^2 = j^2 = k^2 = ijk = -1.$$

(a) Show that $ij = k$ and $jk = i$.

(b) Show that $ij = -ji$.