TA's solution to 2060B homework 5

p.215 Q2. (4 marks)

We go to show that

$$\overline{\int_0^1} h \geq 1, \quad \text{while} \quad \underline{\int_0^1} h \leq 0.$$

Let $P : 0 = x_0 < \ldots < x_n = 1$ be a partition on $[0, 1]$. Since rational numbers are dense in $\mathbb{R}$, so $\forall [x_i, x_{i+1}]$, $\exists t_i \in [x_i, x_{i+1}] \cap \mathbb{Q}$. Then

$$\begin{aligned} U(h, P) &= \sum_i \sup\{h(x) : x \in [x_i, x_{i+1}]\} \cdot [x_{i+1} - x_i] \\ &\geq \sum_i h(t_i) \cdot [x_{i+1} - x_i] \\ &= \sum_i (t_i + 1) \cdot [x_{i+1} - x_i] \\ &\geq \sum_i [x_{i+1} - x_i] = 1. \end{aligned}$$

On the other hand, since irrational numbers are also dense in $\mathbb{R}$, so $\forall [x_i, x_{i+1}]$, $\exists t'_i \in [x_i, x_{i+1}] \setminus \mathbb{Q}$. Hence

$$\begin{aligned} L(h, P) &= \sum_i \inf\{h(x) : x \in [x_i, x_{i+1}]\} \cdot [x_{i+1} - x_i] \\ &\leq \sum_i h(t'_i) \cdot [x_{i+1} - x_i] = 0. \end{aligned}$$

Since P is arbitrary, we have

$$\overline{\int_0^1} h = \inf\{U(h, Q) : Q \text{ partition on } [0, 1]\} \geq 1$$

and

$$\underline{\int_0^1} h = \sup\{L(h, Q) : Q \text{ partition on } [0, 1]\} \leq 0,$$

which was to be demonstrated.

p.215 Q8. (4 marks)

Suppose $f(x_0) > 0$ for some $x_0 \in [a,b]$. Then by the continuity of f, $\exists \delta > 0$ s.t. $f(x) > f(x_0)/2\ \forall x \in (x_0 - \delta, x_0 + \delta) \cap [a,b] := I$. Note that I is always an interval of positive length, regardless of whether $x_0 \in (a,b)$ or $x_0 \in \{a,b\}$. Denote its length by ℓ, then

$$\int_a^b f \geq^* \frac{f(x_0)}{2} \cdot \ell > 0,$$

which is a contradiction[†].

Remark:

If you have written down something like:

"$\exists \delta > 0$ s.t. $f(x) > \frac{f(c)}{2}\ \forall x \in (c-\delta, c+\delta) \subseteq [a,b]$"

in the homework, please see if the following expression is better:

"$\exists \delta > 0$ s.t. $(c-\delta, c+\delta) \subseteq [a,b]$ and $f(x) > \frac{f(c)}{2}\ \forall x \in (c-\delta, c+\delta)$".

p.215 Q9. (2 marks)

Define $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$$

Since g equals the zero function on $[0,1]$ except for a finite no. of points, so it is in $\mathcal{R}[0,1]$ and $\int g = 0$[‡]. Now $g \geq 0$ but it is not identically zero.

[*] This inequality can be justified by e.g. textbook 7.1.5 Theorem (c): consider a step function on $[a,b]$ which is zero outside I and takes the constant value $f(x_0)/2$ inside I.

[†] If you think this solution is stereotyped, then the following approach (provided by one of the students) may be more interesting: consider the function $F(t) := \int_a^t f$, and the Fundamental Theorem of Calculus. Note the monotonicity of F. Apply the mean value theorem to F on a subinterval of I to get a contradiction.

[‡] This is textbook 7.1.3 Theorem.