TA's solution to 2060B homework 4

p.207 Q6. (3 + 3 marks)

Fix $a_0 \in \mathbb{R}$ and define $g : [0, 2] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ a_0 & \text{if } x = 1 \\ 1 & \text{if } 1 < x \le 2. \end{cases}$$

We go to show that regardless of which value a_0 is, we always have $g \in \mathcal{R}[0, 2]$ and $\int_0^2 g = 3$.

Let $A := \max\{2, |a_0|\}$. For the partition $P_\varepsilon : 0 < 1 - \varepsilon < 1 + \varepsilon < 2$ (where $\varepsilon > 0$ is small enough so that P_ε makes sense), we have

$$\begin{aligned} U(g, P_\varepsilon) &= 2 \cdot [(1 - \varepsilon) - 0] + \max\{2, a_0, 1\} \cdot [(1 + \varepsilon) - (1 - \varepsilon)] + 1 \cdot [2 - (1 + \varepsilon)] \\ &\le 2 \cdot [(1 - \varepsilon) - 0] + A \cdot [(1 + \varepsilon) - (1 - \varepsilon)] + 1 \cdot [2 - (1 + \varepsilon)] \\ &= 3(1 - \varepsilon) + 2A\varepsilon \\ &= 3 + \varepsilon \cdot (2A - 3). \end{aligned}$$

Therefore, by considering $\varepsilon \downarrow 0$,

$$\overline{\int_a^b} g = \inf\{U(g, P) : P \text{ partition on } [0, 2]\} \le 3.$$

On the other hand,

$$\begin{aligned} L(g, P_\varepsilon) &= 2 \cdot [(1 - \varepsilon) - 0] + \min\{2, a_0, 1\} \cdot [(1 + \varepsilon) - (1 - \varepsilon)] + 1 \cdot [2 - (1 + \varepsilon)] \\ &\ge 2 \cdot [(1 - \varepsilon) - 0] - A \cdot [(1 + \varepsilon) - (1 - \varepsilon)] + 1 \cdot [2 - (1 + \varepsilon)] \\ &= 3(1 - \varepsilon) - 2A\varepsilon \\ &= 3 - \varepsilon \cdot (2A + 3). \end{aligned}$$

Therefore,

$$\underline{\int_a^b} g = \sup\{L(g, P) : P \text{ partition on } [0, 2]\} \ge 3.$$

This allows us to conclude that $g \in \mathcal{R}[0, 2]$ and $\int_0^2 g = 3$.

p.207 Q7. (4 marks)

The case $n = 1$ follows exactly from 7.1.5 Theorem (a). Suppose the statement is true for $n = \ell$. i.e. $g := \sum_{i=1}^{\ell} k_i f_i \in \mathcal{R}[a,b]$, and $\int_a^b g = \sum_{i=1}^{\ell} k_i \int_a^b f_i$.

By 7.1.5 Theorem (a), $k_{\ell+1} f_{\ell+1} \in \mathcal{R}[a,b]$. Hence by 7.1.5 Theorem (b), $g + k_{\ell+1} f_{\ell+1} \in \mathcal{R}[a,b]$, and $\int_a^b (g + k_{\ell+1} f_{\ell+1}) = \int_a^b g + \int_a^b k_{\ell+1} f_{\ell+1}$. As $\int_a^b k_{\ell+1} f_{\ell+1} = k_{\ell+1} \int_a^b f_{\ell+1}$ by 7.1.5 Theorem (a), so

$$\int_a^b \sum_{i=1}^{\ell+1} k_i f_i = \int_a^b (g + k_{\ell+1} f_{\ell+1}) = \int_a^b g + k_{\ell+1} \int_a^b f_{\ell+1} = \sum_{i=1}^{\ell+1} k_i \int_a^b f_i,$$

where we have used the induction hypothesis in the last step.

By the principle of mathematical induction, the statement is true for all $n \in \mathbb{N}$.