Solution to Assignment 4

Supplementary Problems

1. Determine which of the following functions are convex/strictly convex:

   (a) $f_1(x) = x^p,\ x \in (0, \infty)$.

   (b) $f_2(x) = x^x\ , x \in (0, \infty)$.

   (c) $f_3(x) = \tan x\ , x \in (-\pi/2, \pi/2)$.

   (d) $f_4(x) = x \log x,\ x \in (0, \infty)$.

   (e) $f_5(x) = (1 + \sqrt{x})^{-1}, \quad x \in (-1, \infty)$.

   Solution. (a) $f_1''(x) = p(p-1)x^{p-2} > 0$ on $(0, \infty)$, so it is strictly convex when $p > 1$ or $p < 0$, convex at $p = 0, 1$, and strictly concave when $p \in (0, 1)$. (A function is concave (resp. strictly concave) if its negative is convex (resp. strictly convex).)

   (b) $f_2''(x) = x^x(1 + \log x)^2 + x^{x-1} > 0$, so it is strictly convex.

   (c) $f_3''(x) = 2\sec^2 x \tan x$ is positive on $(0, \pi/2)$ but negative on $(-\pi, 0)$, so it is not strictly convex on $(-\pi/2, \pi/2)$.

   (d) $f_4''(x) = 1/x > 0$ on $(0, \infty)$, so it is strictly convex.

2. Let f and g be two convex functions defined on I. Show that the function $h(x) = \max\{f(x), g(x)\}$ is convex. Is the function $j(x) = \min\{f(x), g(x)\}$ convex?

   Solution. Let $x, y \in I$ and $\lambda \in n[0, 1]$, we have
   $$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y) \le (1-\lambda)h(x) + \lambda h(y),$$
   and
   $$g((1-\lambda)x + \lambda y) \le (1-\lambda)g(x) + \lambda g(y) \le (1-\lambda)h(x) + \lambda h(y),$$
   and the result follows. The min function is in general not convex. For instance you take $f(x) = x^2$ and $g(x) = (x-1)^2$. Then $j = \min\{f, g\}$ is not convex. Plot the graphs to convince yourself.

3. Give an example to show that the product of two strictly convex functions may not be convex. How about the composite of two strictly convex functions?

   Solution. Take $f(x) = x^{3/2},\ g(x) = x^{-1}$ on $(0, 1)$. Then
   $$f''(x) = \frac{3}{2}\frac{1}{2}x^{-1/2} > 0\ ,\ g'(x) = -x^{-2},\ g''(x) = 2x^{-3} > 0$$
   on $(0, 1)$. Hence $f,\ g$ are convex. Now $(fg)(x) = \sqrt{x}$ on $(0, 1)$ is not convex. (In fact, it is strictly concave as $(fg)''(x) = -x^{-3/2}/2 < 0$.)

   If we consider the composition of two twice differentiable functions $F(G(x))$. We have
   $$\frac{d^2}{dx^2}F(G(x)) = F'(G(x))G''(x) + F''(G(x))G'(x)^2\ .$$
   We see that it is convex provided F and G are both convex and F is increasing. In general, only the convexity of both functions is not sufficient. For instance, consider the function $h(x) = e^{-x^2}$ which is the composition of two strictly convex functions $G(x) = x^2$ and $F(y) = e^{-y}$ but
   $$h''(x) = 2(2x^2 - 1)e^{-x^2}, \quad x \in (-\infty, \infty)\ ,$$
   and is negative, say, at $x = 0$.

4. Let f be a convex function on (a,b) whose inverse exists. Is the inverse function convex?

   No. For instance, $f(x)=x^2$ is strictly convex and strictly increasing on $(0,\infty)$. Its inverse exists and is equal to $g(x)=\sqrt{x},\ x\in(0,\infty)$. However, g is strictly concave. In general, from the relation
$$f^{-1}(y)=\frac{1}{f'(x)}\ ,\quad y=f(x)\ ,$$
   we see that the slope of the inverse function is decreasing whenever the slope of f is increasing. Therefore, the inverse function of a differentiable convex function with non-vanishing derivative is strictly concave. In general, it can be shown that the inverse of a convex function is concave.

5. Let f be a continuous function on (a,b) satisfying
$$f\left(\frac{x+y}{2}\right)\le\frac{1}{2}\Big(f(x)+f(y)\Big),\quad \forall x,y\in(a,b).$$
   Show that f is convex. Suggestion: Show
$$f\left(\frac{x_1+\cdots+x_n}{n}\right)\le\frac{f(x_1)+\cdots+f(x_n)}{n}\ ,$$
   for $n=2^m$.

   Solution. Let us show it holds for $n=2^m$ first. Use induction on m. When $m=1$, done by assumption. Assuming it holds at m, we show it for $m+1$. For $x_1,\cdots,x_{2^{m+1}}$, we have
$$\frac{x_1+\cdots+x_{2^{m+1}}}{2^{m+1}}=\frac{1}{2}\frac{x_1+\cdots+x_{2^m}}{2^m}+\frac{1}{2}\frac{x_{2^m+1}+\cdots+x_{2^{m+1}}}{2^m}\ .$$
   Therefore, first by assumption and then by induction hypothesis
$$\begin{aligned}f\left(\frac{x_1+\cdots+x_{2^{m+1}}}{2^{m+1}}\right)&\le\frac{1}{2}f\left(\frac{x_1+\cdots+x_{2^m}}{2^m}\right)+\frac{1}{2}f\left(\frac{x_{2^m+1}+\cdots+x_{2^{m+1}}}{2^m}\right)\\&\le\frac{1}{2}\left(\frac{f(x_1)+\cdots+f(x_{2^m})}{2^m}+\frac{f(x_{2^m+1})\cdots+f(x_{2^{m+1}})}{2^m}\right)\\&=\frac{f(x_1)+\cdots+f(x_{2^{m+1}})}{2^{m+1}}\ .\end{aligned}$$
   After we have proved the inequality for 2^m, we "collapse" it by taking $x=x_1=\cdots=x_n$ and $y=x_{n+1}=\cdots=x_{2^m}$ to get
$$f\left(\frac{n}{2^m}x+\left(1-\frac{n}{2^m}\right)y\right)\le\frac{n}{2^m}f(x)+\left(1-\frac{n}{2^m}\right)f(y)\ ,$$
   so the inequality holds for all λ of the form $n/2^m, 0\le n\le 2^m$. Since every $\lambda\in(0,1)$ can be approximated by such rational numbers, by the continuity of f we have
$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)\ ,$$
   so f is convex.

   We point out that there exist discontinuous functions satisfying this "mean convex property" but is not convex. Google for it in case you are interested in such pathological example.

6. Let f be differentiable on $[a, b]$. Show that it is convex if and only if

$$f(y) - f(x) \geq f'(x)(y - x), \qquad \forall x, y \in [a, b].$$

What is the geometric meaning of this inequality?

Solution. Suppose f is convex, then by Theorem 1.5 of Notes 1, we have f' is increasing function. Let $x \neq y \in [a, b]$. By Mean-Value Theorem, $\exists \xi$ in between x and y such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

Hence

$$\frac{f(y) - f(x)}{y - x} = f'(\xi) = \begin{cases} \geq f(x), & \text{if } x < y. \\ \leq f(x), & \text{if } x > y. \end{cases}$$

Suppose $f(y) - f(x) \geq f'(x)(y - x), \quad \forall x, y \in [a, b]$. We attempt to show that f' is increasing. Let $y > x$, by our assumption, we have

$$f(y) - f(x) \geq f'(x)(y - x)$$

and

$$f(x) - f(y) \geq f'(y)(x - y)$$

which imply

$$f'(y) \geq \frac{f(x) - f(y)}{x - y} = \frac{f(y) - f(x)}{y - x} \geq f'(x).$$

Therefore, f' is increasing. Again by Theorem 1.5 of Notes 1, f is convex on $[a, b]$.

The geometric meaning is, a differentiable function is convex if and only if its tangent line at any point always lies below the graph of the function.

7. Establish the following two inequalities

(a)

$$\sin x + \sin y + \sin z \leq \frac{3\sqrt{3}}{2} .$$

(b)

$$\sin x \ \sin y \ \sin z \leq \frac{3\sqrt{3}}{8} .$$

(c)

$$\frac{1}{3}\left(\frac{1}{\sin x} + \frac{1}{\sin y} + \frac{1}{\sin z}\right) \geq \frac{2}{\sqrt{3}} .$$

Here x, y, z are the three interior angles of a triangle.

Solution.

(a) The sine function is concave on $[0, \pi]$. Therefore,

$$\sin\frac{\pi}{3} = \sin\frac{x + y + z}{3} \geq \frac{\sin x + \sin y + \sin z}{3} ,$$

implies the first inequality.

(b) Next, the function $\log \sin x$ is concave everywhere (actually its second derivative is equal to $-1/\sin x^2 < 0$.) Therefore,

$$\frac{\log \sin x + \log \sin y + \log \sin z}{3} \leq \log \sin \left(\frac{x+y+z}{3} \right) = \log \sin \frac{\pi}{3} \ ,$$

implies the second inequality.

(c) Use the concavity of the function $1/\sin x$.

Since these functions are strictly concave, the inequality signs are strict unless $x = y = z$. Using $x + y + z = \pi$, conclude that equality signs hold in these three inequalities if and only if $x = y = z = \pi/3$, that is, for an equilateral triangle.

8. Establish the inequality

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3} \right)^{a+b+c} \ , \quad a, b, c > 0 \ .$$

Hint: Use of one the functions in (1).

Solution. Take log of both sides and apply Jensen's Inequality to $x \log x$.