THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 6 (October 19)

The following were discussed in the tutorial this week:

1. Let (a_n) be a bounded sequence of real numbers. Show that
   (a) (i) $\overline{\lim_n} a_n \leq \alpha$ if and only if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $a_n < \alpha + \varepsilon$ for all $n \geq N$.
   (ii) $\overline{\lim_n} a_n \geq \alpha$ if and only if for any $\varepsilon > 0$, for all $n \in \mathbb{N}$, there is $k \geq n$ such that $a_k > \alpha - \varepsilon$.
   (b) (i) $\underline{\lim_n} a_n \geq \beta$ if and only if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $a_n > \beta - \varepsilon$ for all $n \geq N$.
   (ii) $\underline{\lim_n} a_n \leq \beta$ if and only if for any $\varepsilon > 0$, for all $n \in \mathbb{N}$, there is $k \geq n$ such that $a_k < \beta + \varepsilon$.

2. (Stolz-Cesàro Theorem (∞/∞ case)) Let (a_n) and (b_n) be two sequences of real numbers such that (b_n) is strictly increasing and $\lim_n b_n = \infty$. Show that

$$\underline{\lim_n} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \leq \underline{\lim_n} \frac{a_n}{b_n} \leq \overline{\lim_n} \frac{a_n}{b_n} \leq \overline{\lim_n} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}.$$

3. Use Stolz-Cesàro Theorem to evaluate the following limits.
   $\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}}$, where $p \in \mathbb{N}, p \neq 1$

4. Let (x_n) be a **contractive sequence** of rate $r \in (0, 1)$, that is

$$|x_{n+1} - x_n| \leq r|x_n - x_{n-1}| \qquad \text{for } n \geq 1.$$

   (a) Show that

$$|x_{n+k} - x_n| \leq \frac{r^{n-1}}{1-r}|x_2 - x_1| \qquad \text{for } n, k \geq 1.$$

   (b) Hence, show that (x_n) is convergent.

5. Let (x_n) be defined by

$$\begin{cases} x_1 = 1, \quad x_2 = 2; \\ x_n = \frac{1}{3}(2x_{n-1} + x_{n-2}) \quad \text{for } n > 2. \end{cases}$$

   Show that (x_n) converges and find its limit.

6. (Root Test for Series) Let (x_n) be a real sequence. Set

$$r := \overline{\lim_n} |x_n|^{1/n}.$$

(a) If $r < 1$, then the series $\sum x_n$ is absolutely convergent.

(b) If $r > 1$, then the series $\sum x_n$ is divergent.

7. Use the definition of limit to show that $\lim\limits_{x \to -1} \dfrac{x^2 + 2x + 4}{x + 2} = 3$.