TA's selected solution to 2050B Test 2

1. (a). (x_n) converges to ℓ if and only if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \quad \text{we have } |x_n - \ell| < \varepsilon.$$

(x_n) does not converge to ℓ if and only if

$$\exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \; \exists n \geq N \text{ s.t. } |x_n - \ell| \geq \varepsilon.$$

Steps:

$$\begin{aligned}
&\textbf{Not } (\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \quad \text{we have } |x_n - \ell| < \varepsilon) \\
&= \exists \varepsilon > 0 \text{ s.t. } \textbf{Not } (\exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \quad \text{we have } |x_n - \ell| < \varepsilon) \\
&= \exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N} \; \textbf{Not } (\forall n \geq N, \quad \text{we have } |x_n - \ell| < \varepsilon) \\
&= \exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N} \; \exists n \geq N \text{ s.t. } \textbf{Not } (\text{we have } |x_n - \ell| < \varepsilon) \\
&= \exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N} \; \exists n \geq N \text{ s.t. } |x_n - \ell| \geq \varepsilon
\end{aligned}$$

(b). $f(x)$ converges to ℓ as $x \to x_0$ if and only if:
Given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$ with $0 < |x - x_0| < \delta$, we have $|f(x) - \ell| < \varepsilon$.*

$f(x)$ does not converge to ℓ as $x \to x_0$ if and only if:
There exists $\varepsilon > 0$ such that for any $\delta > 0$, we can find some $x \in D$ with $0 < |x - x_0| < \delta$ such that $|f(x) - \ell| \geq \varepsilon$.

3. Please refer to Prof. Ng's lecture notes in the course web page.

*There are 3 conditions on x. The first is $|x - x_0| < \delta$ which means x is close to x_0. The second is $x \in D$ so that $f(x)$ is defined. The third is $x \neq x_0$ (i.e. $|x - x_0| > 0$); think about the case $x_0 := 1$ and $f(x) := \begin{cases} x \text{ if } x \neq 1 \\ 2 \text{ if } x = 1. \end{cases}$