MATH2050A: Analysis I (2018 1st term)

1 Compact Sets in $\mathbb{R}$

Throughout this section, A always denotes a subset of $\mathbb{R}$.

We say that a sequence (x_n) in A is *convergent in A* if there is an element "$a \in A$" such that for every $\varepsilon > 0$, there is a positive integer N so that $|x_n - a| < \varepsilon$ whenever $n \geq N$. For example, the sequence $(1/n)$ is convergent in $\mathbb{R}$ but it is not convergent in $(0, 1]$.

When we consider the case $A = \mathbb{R}$, it is simply to say that a sequence is convergent if its limit exists.

On the other hand, a subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, that is, such sequence (n_k) can be viewed as a strictly increasing function $\mathbf{n} : k \in \{1, 2, ..\} \mapsto n_k \in \{1, 2, ...\}$.
In this case, note that for each positive integer N, there is $K \in \mathbb{N}$ such that $n_K \geq N$ and thus we have $n_k \geq N$ for all $k \geq K$.

Proposition 1.1 *Let (x_n) be a sequence in $\mathbb{R}$. Then the following statements are equivalent.*

(i) (x_n) is convergent.

(ii) Any subsequence (x_{n_k}) of (x_n) converges to the same limit.

(iii) Any subsequence (x_{n_k}) of (x_n) is convergent.

Proof: Part $(ii) \Rightarrow (i)$ is clear because the sequence (x_n) is also a subsequence of itself.
For the Part $(i) \Rightarrow (ii)$, assume that $\lim x_n = a$ exists. Let (x_{n_k}) be a subsequence of (x_n). We claim that $\lim x_{n_k} = a$. Let $\varepsilon > 0$. In fact, since $\lim x_n = a$, there is a positive integer N such that $|a - x_n| < \varepsilon$ for all $n \geq N$. Notice that by the definition of a subsequence, there is a positive integer K such that $n_k \geq N$ for all $k \geq K$. So, we see that $|a - x_{n_k}| < \varepsilon$ for all $k \geq K$. Thus we have $\lim_{k\to\infty} x_{n_k} = a$.
Part $(ii) \Rightarrow (iii)$ is clear.
It remains to show Part $(iii) \Rightarrow (ii)$. Suppose that there are two subsequences $(x_{n_i})_{i=1}^{\infty}$ and $(x_{m_i})_{i=1}^{\infty}$ converge to distinct limits. Now put $k_1 := n_1$. Choose $m_{i'}$ such that $n_1 < m_{i'}$ and then put $k_2 := m_{i'}$. Then we choose n_i such that $k_2 < n_i$ and put k_3 for such n_i. To repeat the same step, we can get a subsequence $(x_{k_i})_{i=1}^{\infty}$ of (x_n) such that $x_{k_{2i}} = x_{n_{i'}}$ for some $n_{i'}$ and $x_{k_{2i-1}} = x_{m_{j'}}$ for some $m_{j'}$. Since by the assumption $\lim_i x_{n_i} \neq \lim_i x_{m_i}$, $\lim_i x_{k_i}$ does not exist which leads to a contradiction.
The proof is finished. $\square$

Definition 1.2 We say that A is a *closed subset of* $\mathbb{R}$ *(or closed set for simply)* if it satisfies the condition: if (x_n) is a sequence in A and the limit $\lim x_n$ exists, then $\lim x_n \in A$.

Example 1.3 (i) The empty set is a closed subset of $\mathbb{R}$.

(ii) The union of finitely closed subintervals is a closed set.

(iii) The set of integers $\mathbb{Z}$ is a closed set.

(iv) The set of all rational number $\mathbb{Q}$ is not a closed set.

The following Lemma can be directly shown by the definition, so, the proof is omitted here.

Lemma 1.4 *Let A be a subset of $\mathbb{R}$. The following statements are equivalent.*

(i) A is closed.

(ii) For each element $x \in \mathbb{R} \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$.

We now recall the following important theorem in $\mathbb{R}$ (see [1, Theorem 3.4.8]).

Theorem 1.5 Bolzano-Weierstrass Theorem *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Definition 1.6 A subset A of $\mathbb{R}$ is said to be compact if for every sequence in A has a convergent subsequence in A, that is, if (x_n) is a sequence in A, then it has a subsequence (x_{n_k}) that converges to some element in A.

Example 1.7 (i) Every finite subset is compact.

(ii) Every closed and bounded interval is compact.
In fact, if (x_n) is any sequence in a closed and bounded interval $[a, b]$, then (x_n) is bounded. Then by Bolzano-Weierstrass Theorem (see [1, Theorem 3.4.8]), (x_n) has a convergent subsequence (x_{n_k}). Notice that since $a \leq x_{n_k} \leq b$ for all k, then $a \leq \lim_k x_{n_k} \leq b$, and thus $\lim_k x_{n_k} \in [a, b]$. Therefore A is compact.

(iii) $(0, 1]$ is not compact. In fact, if we consider $x_n = 1/n$, then (x_n) is a sequence in $(0, 1]$ but it has no convergent subsequence with the limit sitting in $(0, 1]$.

Theorem 1.8 *Let A be a subset of $\mathbb{R}$. Then A is compact if and only if A is a closed and bounded subset of $\mathbb{R}$.*

Proof: For showing the necessary condition, assume that A is compact. We first claim that A is bounded. Suppose not. We suppose that A is unbounded. If we fix an element $x_1 \in A$, then there is $x_2 \in A$ such that $|x_1 - x_2| > 1$. Using the unboundedness of A, we can find an element x_3 in A such that $|x_3 - x_k| > 1$ for $k = 1, 2$. To repeat the same step, we can find a sequence (x_n) in A such that $|x_n - x_m| > 1$ for $n \neq m$. Thus A has no convergent subsequence. Thus A must be bounded.
Next, we want to show that A is closed in $\mathbb{R}$. Let (x_n) be a sequence in A and it is convergent.

It needs to show that $\lim_n x_n \in A$. Note that since A is compact, (x_n) has a convergent subsequence (x_{n_k}) such that $\lim_k x_{n_k} \in A$. Then $\lim_n x_n = \lim_k x_{n_k} \in A$. Thus, A is closed. Conversely, assume that A is closed and bounded. Let (x_n) be a sequence in A and thus (x_n) is a bounded sequence in $\mathbb{R}$. Then by the Bolzano- Weierstrass Theorem, (x_n) has a subsequence (x_{n_k}) which is convergent in $\mathbb{R}$. Since A is closed, $\lim_k x_{n_k} \in A$. Therefore, A is compact. $\square$

Definition 1.9 A subset A of $\mathbb{R}$ is said to have *Heine-Borel property* if for any open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of A, that is, each J_α is an open interval and

$$A \subseteq \bigcup_{\alpha\in\Lambda} J_\alpha,$$

we can find finitely many $J_{\alpha_1}, .., J_{\alpha_N}$ such that $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$.

Example 1.10 $(0,1]$ does not have Heine-Borel property. In fact, if we put $J_n = (1/n, 2)$ for $n = 2, 3...$, then $(0,1] \subseteq \bigcup_{n=2}^\infty J_n$, but we cannot find finitely many $J_{n_1}, ..., J_{n_K}$ such that $(0,1] \subseteq J_{n_1} \cup \cdots \cup J_{n_K}$.

Let us first recall one of the important properties of real line.

Theorem 1.11 Nested Intervals Theorem *Let $(I_n := [a_n, b_n])$ be a sequence of closed and bounded intervals. Suppose that it satisfies the following conditions.*

(i) : $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

(ii) : $\lim_n (b_n - a_n) = 0$.

Then there is a unique real number ξ such that $\bigcap_{n=1}^\infty I_n = \{\xi\}$.

Proof: See [1, Theorem 2.5.2, Theorem 2.5.3]. $\square$

Theorem 1.12 (Heine-Borel Theorem) *Every closed and bounded interval $[a, b]$ has Heine-Borel property.*

Proof: Suppose that $[a, b]$ does not have Heine-Borel property. Then there is an open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $[a, b]$ but it it has no finite sub-cover. Let $I_1 := [a_1, b_1] = [a, b]$ and m_1 the mid-point of $[a_1, b_1]$. Then by the assumption, $[a_1, m_1]$ or $[m_1, b_1]$ cannot be covered by finitely many J_α's. We may assume that $[a_1, m_1]$ cannot be covered by finitely many J_α's. Put $I_2 := [a_2, b_2] = [a_1, m_1]$. To repeat the same steps, we can obtain a sequence of closed and bounded intervals $I_n = [a_n, b_n]$ with the following properties:

(a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots$;

(b) $\lim_n (b_n - a_n) = 0$;

(c) each I_n cannot be covered by finitely many J_α's.

Then by the Nested Intervals Theorem, there is an element $\xi \in \bigcap_n I_n$ such that $\lim_n a_n = \lim_n b_n = \xi$. In particular, we have $a = a_1 \leq \xi \leq b_1 = b$. So, there is $\alpha_0 \in \Lambda$ such that $\xi \in J_{\alpha_0}$. Since J_{α_0} is open, there is $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. On the other hand, there is $N \in \mathbb{N}$ such that a_N and b_N in $(\xi - \varepsilon, \xi + \varepsilon)$ because $\lim_n a_n = \lim_n b_n = \xi$. Thus we have $I_N = [a_N, b_N] \subseteq (\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. It contradicts to the Property (c) above. The proof is finished. $\square$

Theorem 1.13 *Let A be a subset of $\mathbb{R}$. The following statements are equivalent.*

(i) A has Heine-Borel property.

(ii) A is compact.

(iii) A is closed and bounded.

Proof: The result is shown by the following path $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.
Part $(i) \Rightarrow (ii)$ will be shown by contradiction. Suppose that A has Heine-Borel property but it is not compact. Then there is a sequence (x_n) in A such that (x_n) has no convergent subsequent in A. Put $F = \{x_n : n = 1, 2, ...\}$. Then F is infinite and hence for each element $a \in A$, there is $\delta_a > 0$ such that $(a - \delta_a, a + \delta_a) \cap F$ is finite. Indeed, if there is an element $a \in A$ such that $(a - \delta, a + \delta) \cap F$ is infinite for all $\delta > 0$, then (x_n) has a convergent subsequence with the limit $a \in A$. Let $J_a := (a - \delta_a, a + \delta_1)$. On the other hand, we have $A \subseteq \bigcup_{a \in A} J_a$. Then by the Heine-Borel property of A, we can find finitely many $a_1, ..., a_N$ such that $A \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. In particular, we have $F \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. Then by the choice of J_a's, A must be finite. This leads to a contradiction. Therefore, A is compact.

Part $(ii) \Rightarrow (iii)$ follows from Theorem 1.8 at once.

It remains to show $(iii) \Rightarrow (i)$. Suppose that A is closed and bounded. Then we can find a closed and bounded interval $[a, b]$ such that $A \subseteq [a, b]$. Now let $\{J_\alpha\}_{\alpha \in \Lambda}$ be an open intervals cover of A. Notice that for each element $x \in [a, b] \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$ since A is closed. If we put $I_x = (x - \delta_x, x + \delta_x)$ for $x \in [a, b] \setminus A$, then we have

$$[a, b] \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha \cup \bigcup_{x \in [a,b] \setminus A} I_x.$$

Using the Heine-Borel Theorem 1.12, we can find finitely many J_α's and I_x's, say $J_{\alpha_1}, ..., J_{\alpha_N}$ and $I_{x_1}, ..., I_{x_K}$, such that $A \subseteq [a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N} \cup I_{x_1} \cup \cdots \cup I_{x_K}$. Note that $I_x \cap A = \emptyset$ for each $x \in [a, b] \setminus A$ by the choice of I_x. Therefore, we have $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$ and hence A has Heine-Borel property.
The proof is finished. $\square$

2 Limits of functions

Throughout this, section, let A be a subset of $\mathbb{R}$.

Definition 2.1 A point $c \in \mathbb{R}$ is called a limit point (or cluster point) of A if for each $r > 0$, there is an element $x \in A$ such that $0 < |x - c| < r$, that is, $\big((c - r, c + r) \setminus \{c\}\big) \cap A \neq \emptyset$. From now, let $D(A)$ be the set of all limit points of A.

Example 2.2 (i) $D(\mathbb{N}) = \emptyset$.

(ii) If we let $A = [0, 1) \cup \{2\}$, then $D(A) = [0, 1]$.

(iii) $D(\mathbb{Q}) = \mathbb{R}$ (why?)

The following result can be shown by the definition directly.

Proposition 2.3 *Using the notation as above, then A is a closed subset of $\mathbb{R}$ if and only of $D(A) \subseteq A$.*
Consequently, if $D(A) = \emptyset$, then A is closed in $\mathbb{R}$ automatically.

Theorem 2.4 *Let f be a real-valued function defined a non-empty subset A of $\mathbb{R}$ and let c be a limit point of A. Then the followings are equivalent.*

(i) $\lim_{x\to c} f(x)$ *exists.*

(ii) For each sequence (x_n) in A with $\lim_n x_n = c$ and $x_n \neq c$ for all n, the sequence $(f(x_n))$ converges to the same limit.

(iii) For each sequence (x_n) in A with $\lim_n x_n = c$ and $x_n \neq c$ for all n, the sequence $(f(x_n))$ is convergent.

(iv) For each $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ satisfy $0 < |x - c| < \delta$ and $0 < |y - c| < \delta$.

In this case $\lim_{x\to c} f(x) = \lim_n f(x_n)$ whenever a sequence (x_n) in A with $\lim_n x_n = c$ and $x_n \neq c$ for all n.

Proof: For $(i) \Rightarrow (ii)$, suppose that $L := \lim_{x\to c} f(x)$ exists. Then for each $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - L| < \varepsilon$ as $x \in A$ with $0 < |x - c| < \delta$. So, if (x_n) in A with $\lim_n x_n = c$ and $x_n \neq c$ for all n, then there is N such that $0 < |x_n - c| < \delta$ for all $n \geq N$. This gives $|f(x_n) - L| < \varepsilon$ for all $n \geq N$ and thus, $\lim f(x_n) = L$.
$(ii) \Rightarrow (iii)$ is clear.
For showing $(iii) \Rightarrow (iv)$, suppose that (iv) is not true. Then there is $\varepsilon > 0$ such that for each $\delta > 0$, there exist x and y in A with $0 < |x - c| < \delta$ and $0 < |y - c| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. By considering $\delta = 1/n$ for $n = 1, 2, ..$, then we can find sequences (x_n) and (y_n) in A such that $x_n \neq c \neq y_n$ with $\lim x_n = \lim y_n = c$ but $|f(x_n) - f(y_n)| \geq \varepsilon$ for all $n = 1, 2....$ Now if we put $z_{2n} := x_n$ and $z_{2n-1} := y_n$, then $z_n \neq c$ for all n and $\lim z_n = c$ but $\lim_n f(z_n)$ does not exist because $f(z_n)$ is not a Cauchy sequence. Hence, (iv) does not hold and thus, we have $(iii) \Rightarrow (iv)$.
Finally, we want to show the implication $(iv) \Rightarrow (i)$. Notice that since c is a limit point of A,

we can find a sequence in $A \setminus \{c\}$ such that $\lim_n x_n = c$. Then the condition (iv) tells us that the sequence $(f(x_n))$ is a Cauchy sequence and hence, $L := \lim f(x_n)$ exists.
The part (i) follows if we can show $L = \lim_{x\to c} f(x)$. Indeed, for each $\varepsilon > 0$, let $\delta > 0$ be found as in the condition (iv). Since $\lim x_n = c$ with $x_n \neq c$ and $L := \lim f(x_n)$, we can find a positive integer N such that $0 < |x_N - c| < \delta$ and $|f(x_N) - L| < \varepsilon$. Then by the choice of δ, if $x \in A$ with $0 < |x - c| < \delta$, we have $|f(x) - f(x_N)| < \varepsilon$. This implies that $|f(x) - L| \leq |f(x) - f(x_N)| + |f(x_N) - L| < 2\varepsilon$ for all $x \in A$ with $0 < |x - c| < \delta$. Hence, $L = \lim_{x\to c} f(x)$.
The last assertion follows from the proof of $(i) \Rightarrow (ii)$ above.
The proof is finished. $\square$

Definition 2.5 Let A be an unbounded above subset of $\mathbb{R}$ and f be a function defined on A.

(i) We say that a sequence (x_n) in $\mathbb{R}$ tends to infinity, write $\lim x_n = \infty$, if for each $M > 0$, there is a positive integer N such that $x_n > M$ for all $n \geq N$. (**NOTE:** the infinity is NOT the limit in this case).

(ii) We say that f converges to a number L as x going to infinity if for each $\varepsilon > 0$, there is $M > 0$, such that $|f(x) - L| < \varepsilon$ whenever $x \in A$ with $x > M$. In this case, write $\lim\limits_{x\to\infty} f(x) = L$.

Similarly, one can define f converges to L as $n \to -\infty$, $L = \lim\limits_{x\to-\infty} f(x)$, when A is not bounded below.

Proposition 2.6 *Using the notation as above, the followings are equivalent.*

(i) $\lim\limits_{x\to\infty} f(x)$ *exists.*

(ii) $(f(x_n))$ *converges to the same limit for every sequence* (x_n) *in* A *with* $\lim x_n = \infty$*.*

(iii) $(f(x_n))$ *is convergent for every sequence* (x_n) *in* A *with* $\lim x_n = \infty$*.*

(iv) For every $\varepsilon > 0$*, there is* $M > 0$ *such that* $|f(x) - f(y)| < \varepsilon$ *whenever* $x, y \in A$ *with* $x, y > M$*.*

In this case $\lim_{x\to\infty} f(x) = \lim_n f(x_n)$ *for every sequence* (x_n) *in* A *with* $\lim x_n = \infty$*.*

Proof: The proof of $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are similar to the proof of Theorem 2.4.
The implication $(ii) \Rightarrow (iii)$ is clear.
It remains to show $(iv) \Rightarrow (i)$. Suppose that (iv) holds. Since A is not bounded above, we can find a sequence (x_n) in A such that $\lim x_n = \infty$. By considering $\varepsilon = 1$ in the condition (iv), there is $M_1 > 0$ such that $|f(x) - f(y)| < 1$ for all $x, y \in A$ with $x, y > M_1$. Since $\lim x_n = \infty$, we can find a positive integer N_1 such that $x_n > M_1$ for all $n \geq N_1$. This implies that $|f(x_n) - f(x_{N_1})| < 1$ for all $n \geq N_1$ and thus, $|f(x_n)| < |f(x_{N_1})| + 1$ all $n \geq N_1$. So, $(f(x_n))$ is a bounded sequence. The Bolzano-Weierstrass Theorem tells us that there is a convergent subsequence $(f(x_{n_k}))$ of $f(x_n)$. Put $L := \lim_k f(x_{n_k})$. The implication $(iv) \Rightarrow (i)$ follows from $\lim_{x\to\infty} f(x) = L$. In fact, let $\varepsilon > 0$ and let M be a positive number as found in the condition (iv). Notice that since $\lim_n x_n = \infty$, we also have $\lim_k x_{n_k} = \infty$. Thus, we can

choose a positive integer K large enough so that $|L - f(x_{n_K})| < \varepsilon$ and $x_{n_K} > M$. Hence, if $x > M$, we have

$$|f(x) - L| \le |f(x) - f(x_{n_K})| + |f(x_{n_K}) - L| < 2\varepsilon.$$

So, $\lim_{x\to\infty} f(x) = L$ as required.
The last assertion follows from the proof in $(i) \Rightarrow (ii)$ at once. The proof is finished. $\square$

3 Continuous Functions

Throughout this section, let f be a real-valued function defined on a subset A of $\mathbb{R}$.

Definition 3.1 A function f is said to be continuous at an element a in A if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in A$ and $|x - a| < \delta$.
We say that f is continuous on A if f is continuous at every point in A.

Remark 3.2 If c is an isolated point of A, i.e., there is $r > 0$ such that $(c-r, c+r)\cap A = \{c\}$, then f must be continuous at c. Therefore, if c is a limit point of A and $c \in A$, then f is continuous at c if and only if $\lim_{x\to c} f(x) = f(c)$.

Proposition 3.3 *Assume that f is continuous on A. If A is compact, then the image $f(A) := \{f(x) : x \in A\}$ is bounded. Moreover, there are points z_1 and z_2 in A such that $f(z_1) = \max f(A)$ and $f(z_2) = \min f(A)$.*

Proof: We first claim that the image $f(A)$ is bounded by using the following two different methods.
Method I:
Suppose not. Then for each positive integer n, there exists an element x_n in A such that $|f(x_n)| > n$. Since A is compact, there is a convergent subsequence (x_{n_k}) of (x_n) such that $z := \lim_k x_{n_k} \in A$. Then by the continuity of f, we have $\lim_k f(x_{n_k}) = f(z)$ and thus, $(f(x_{n_k}))$ is a bounded sequence. However, since $|f(x_{n_k})| > n_k$ for all $k = 1, 2, ...$ It leads to a contradiction.
Method II:
Since f is continuous at every point of A, for each element a in A, there is $\delta(a) > 0$ such that $|f(x) - f(a)| < 1$ for all $x \in A$ with $|x - a| < \delta(a)$. Now for each $a \in A$, set $J(a) := (a - \delta(a), a + \delta(a))$. Then we have $|f(x)| < 1 + |f(a)|$ for all $x \in J(a) \cap A$ and the collection $\{J(a) : a \in A\}$ forms an open intervals cover of A, i.e., $A \subseteq \bigcup_{a\in A} J(a)$. Applying the Heine-Borel property of A (see Theorem 1.13), there are finitely many subcovers, $J(a_1), ..., J(a_N)$ of A, that is, $A \subseteq J(a_1) \cup \cdots \cup J(a_N)$. Take $M := \max(1 + |f(a_1)|, ..., 1 + |f(a_N)|)$. So, for each element x in A, we have $x \in J(a_k)$ for some $J(a_K)$. This gives $|f(x)| < 1 + |f(x_K)| \le M$. Hence, the image $f(A)$ is bounded by M.
Next, we show that there is an element $z \in A$ such that $f(z) = \max f(A)$. In fact by the claim above, $L := \sup f(A)$ exists. Notice that for each positive integer n, there is an element $x_n \in A$ such that $L - 1/n < f(x_n) < L + 1/n$. This implies that $\lim_n f(x_n) = L$. On the other hand, by the compactness of A, there exists a convergent subsequence (x_{n_k}) of (x_n) such that $z := \lim_k x_{n_k} \in A$. So, we have $f(z) = \lim_k f(x_{n_k}) = L$ as required because f is continuous at z.

Finally, by considering the function $-f$, one can also find an element z_2 in A such that $f(z_2) = \min f(A)$. The proof is finished. $\square$

Proposition 3.4 *If f is a continuous function defined on a compact set A, then the image $f(A)$ is also a compact set.*

Proof: The result will be shown by the following two methods.
Method I:
By using Theorem 1.13, we need to show that $f(A)$ is a closed and bounded set. Proposition 3.3 tells us that the image $f(A)$ is bounded. It remains to show that $f(A)$ is a closed subset of $\mathbb{R}$, i.e, if $L = \lim f(x_n)$ for a sequence (x_n) in A, we need to show that $L \in f(A)$. In fact, the compactness of A gives a convergent subsequence (x_{n_k}) of (x_n) such that $z := \lim_k x_{n_k} \in A$. Then by the continuity of A, we have $L = \lim_k f(x_{n_k}) = f(z) \in f(A)$ as desired.
Method II: In her, we will make use the Heine-Borel property of A. Let $\{J_i\}_{i \in I}$ be a collection of open intervals of $f(A)$. Then for each element $a \in A$, we have $f(a) \in J_{i(a)}$ for some $i(a) \in I$. Since $J_{i(a)}$ is an open interval, we can find $\varepsilon_a > 0$ such that $(f(a) - \varepsilon_a, f(a) + \varepsilon_a) \subseteq J_{i(a)}$. On the other hand, there is $\delta_a > 0$ such that $|f(x) - f(a)| < \varepsilon_a$ for all $x \in A$ with $|x - a| < \delta$ because f is continuous at a. So, if we put $W_a := (a - \delta_a, a + \delta_a)$, then we have

$$f(W_a \cap A) \subseteq (f(a) - \varepsilon_a, f(a) + \varepsilon_a) \subseteq J_{i(a)}.$$

On the other hand, we have $A \subseteq \bigcup_{a \in A} W_a$. The Heine-Borel property of A implies that there are finitely many $W_{a_1}, ..., W_{a_N}$ such that

$$A \subseteq W_{a_1} \cup \cdots \cup W_{a_N}.$$

Therefore, we have

$$f(A) \subseteq J_{i(a_1)} \cup \cdots \cup J_{i(a_N)}.$$

The proof is finished. $\square$

Example 3.5 By using Proposition 3.4, it is impossible to find a continuous surjection from $[0, 1]$ onto $\mathbb{R}$.

Definition 3.6 Let A and B be non-empty subsets of $\mathbb{R}$. A bijection f from A onto B is called a homeomorphism if f and its inverse function f^{-1} both are continuous.
In this case, A and B are said to be homeomorphic if there exists a homeomorphism between A and B.

Remark 3.7 In general, if f is a continuous bijection from A onto B, it does not imply that its inverse $f^{-1} : B \to A$ is continuous. For example, define a function $f : [0, 1) \cup [2, 3] \to [0, 2]$ by $f(x) := x$ for $x \in [0, 1)$; and $f(x) := x - 1$ for $x \in [2, 3]$. Then f is a continuous bijection from $[0, 1) \cup [2, 3]$ onto $[0, 2]$ but the inverse $f^{-1}(y)$ is discontinuous at $y = 1$.
In fact, the following result tells us that it is impossible to find a homeomorphism between the sets $[0, 1) \cup [2, 3]$ and $[0, 2]$.

Corollary 3.8 *If a set A is homeomorphic to a set B, then A is compact if and only if B is compact too.*

Proof: It follows from Proposition 3.4 at once. $\square$

4 Uniform continuous functions on compact sets

Throughout this section, A always denotes a non-empty subset of $\mathbb{R}$ and f is a function on A.

Definition 4.1 f is said to be uniformly continuous on A if for each $\varepsilon > 0$ there exists $\delta > 0$ (**depends on ε only**) such that $|f(x) - f(y)| < \varepsilon$ whenever x, y in A with $|x - y| < \delta$.

Remark 4.2 (i) By the definition of uniform continuity, a function f is not uniformly continuous on A if there is $\varepsilon > 0$ such that for each $\delta > 0$, we can find some x and x' in A satisfying $|x - x'| < \delta$ but $|f(x) - f(x')| \geq \varepsilon$.

(ii) It is clear that every uniformly continuous function on A is continuous. However, the converse does not hold.

Example 4.3 Let $A := [1, \infty)$.

(i) If $f_1(x) := x$ for all $x \in A$, then f_1 is clearly uniformly continuous on A.

(ii) If $f_2(x) := x^2$ for all $x \in A$, then f_2 is not uniformly continuous on A. In fact, if we let $x_n := n$ and $y_n = n + \frac{1}{n}$ for each positive integer, then $|x_n^2 - y_n^2| = 1 + \frac{1}{n^2}$. So, let $\varepsilon = 1$. Then for any $\delta > 0$, we can choose a positive integer N so that $1/N < \delta$ and thus we have $|x_N - y_N| < \delta$ but $|f_2(x_N) - f_2(y_N)| \geq \varepsilon$.

(iii) If $f_3(x) := \sqrt{x}$, then f_3 is uniformly continuous on A. In fact, it is follows from the simple calculation that

$$|f_3(x) - f_3(y)| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{2}|x - y|$$

for all $x, y \in A$.

Remark 4.4 From Examples 4.3 (i) and (ii), we see that product of uniformly continuous functions need not be uniformly continuous.
On the other hand, notice that the function f_2 in Example 4.3 is a homeomorphism from A onto itself, i.e, f_2 is a bijection, also, f_2 and its inverse f_2^{-1} both are continuous. Indeed, the inverse of f_2 is given by f_3. From Example 4.3 (ii) and (iii), we see that the uniform continuity cannot be preserved for a homeomorphism.

Theorem 4.5 *If f is a continuous function defined on a compact set A, then f is uniformly continuous on A.*

Proof: Recall that a set A is said to be compact if for every sequence (x_n) in A, we can find subsequence (x_{n_k}) that converges to some element in A. This is also equivalent to saying that A has the Heine-Borel property (see Theorem 1.13).
Method I:
Suppose tha4 f is not uniformly continuous on A. Then there is $\varepsilon > 0$ so that for every $\delta > 0$, we can find some elements x and y in A with $|x-y| < \delta$ but $|f(x)-f(y)| \geq \varepsilon$. From this, there exist the sequences (x_n) and (y_n) in A such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. By using

the compactness of A, (x_n) has a subsequence (x_{n_k}) that converges to some element $z \in A$ and hence, $\lim_k y_{n_k} = z$ because $\lim_k (x_{n_k} - y_{n_k}) = 0$. This gives $\lim_k f(x_{n_k}) = \lim_k f(y_{n_k}) = f(z)$ which leads to a contradiction since $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$ for all k.
Method II: Let $\varepsilon > 0$. Since f is continuous on A, for each element $a \in A$, there is $\delta(a) > 0$ such that $|f(x) - f(y)| < \varepsilon/2$ whenever $x \in A$ with $|x - a| < \delta(a)$. Put

$$J(a) := (a - \frac{1}{2}\delta(a), a + \frac{1}{2}\delta(a)).$$

Then the collection $\{J(a) : a \in A\}$ form an open intervals cover of A. Using the Heine-Borel property, there exists finitely many elements $a_1, ..., a_N$ in A such that $A \subseteq J(a_1) \cup \cdots \cup J(a_N)$. Now we can choose a positive number δ such that $0 < \delta < \frac{1}{2}\delta(a_k)$ for all $k = 1, ..., N$. We will show that the positive number δ that we want. In fact, let $x, y \in A$ with $|x - y| < \delta$. Since $A \subseteq J(a_1) \cup \cdots \cup J(a_N)$, we have $x \in J(a_k)$ for some $k = 1, .., N$. Thus, we have $|x - a_k| < \delta < \frac{1}{2}\delta(a_k)$. Also, from this, we see that $|y - a_k| \leq |y - x| + |x - a_k| < \delta + \frac{1}{2}\delta(a_k) < \delta(a_k)$. Then by the definition of $\delta(a_k)$, we have $|f(x) - f(y)| \leq |f(x) - f(a_k)| + |f(a_k) - f(y)| < \varepsilon$. The proof is finished. □

Proposition 4.6 *Let f be a continuous function defined on (a, b). The the followings are equivalent.*

(i) There exists a continuous function $F : [a, b] \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in (a, b)$.

(ii) f is uniformly continuous on (a, b).

(iii) The limits $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ both exist.

In this case, this continuous extension F is uniquely determined by f. In fact, $F(a) = \lim\limits_{x \to a+} f(x)$ and $F(b) = \lim\limits_{x \to b-} f(x)$.

Proof: For $(i) \Rightarrow (ii)$, we assume that (i) holds. Then by Theorem 4.5, F is uniformly continuous on $[a, b]$. This implies that $f = F|_{(a,b)}$ is uniformly continuous on (a, b) at once.
For $(ii) \Rightarrow (iii)$, we are going to show that $\lim\limits_{x \to b-} f(x)$ exists.
It suffices to show that the sequence $(f(x_n))$ converges to the same limit whenever any sequence (x_n) in (a, b) that converges to b.
We first claim that $(f(x_n))$ is a Cauchy sequence for any such sequence (x_n) in (a, b). Let $\varepsilon > 0$. Then by the assumption (ii), there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ as $x, y \in (a, b)$ with $|x - y| < \delta$. Now since $\lim x_n = b$ and thus, (x_n) is a Cauchy sequence, we can find a positive N such that $|x_m - x_n| < \delta$ when $m, n \geq N$. This gives $|f(x_m) - f(x_n)| < \varepsilon$ as $m, n \geq N$. The claim follows and thus, the limit $\lim\limits_{n \to \infty} f(x_n)$ exists.
Next we want to show that if (x_n) and (y_n) both are the sequences in (a, b) that converge to b, then $\lim\limits_{n \to \infty} f(x_n) = \lim\limits_{n \to \infty} f(y_n)$. Let $L = \lim\limits_{n \to \infty} f(x_n)$ and $L' = \lim\limits_{n \to \infty} f(y_n)$. Let $\varepsilon > 0$ and let δ be given by the uniform continuity of f. Since $\lim x_n = \lim y_n$, we can choose a positive integer N large enough so that $|x_N - y_N| < \delta$. Also, such N satisfies $|f(x_N) - L| < \varepsilon$ and $|f(y_N) - L'| < \varepsilon$ because $L = \lim\limits_{n \to \infty} f(x_n)$ and $L' = \lim\limits_{n \to \infty} f(y_n)$. This implies that

$$|L - L'| \leq |L - f(x_N)| + |f(x_N) - f(y_N)| + |f(y_N) - L'| < 3\varepsilon$$

for all $\varepsilon > 0$. So, $L = L'$ and hence, the limit $\lim\limits_{x\to b-} f(x)$ exist.
The proof of the case $\lim\limits_{x\to a+} f(x)$ is similar.
Finally, we show $(iii) \Rightarrow (i)$. Define $F(a) := \lim\limits_{x\to a+} f(x)$; $F(b) := \lim\limits_{x\to b-} f(x)$ and $F(x) := f(x)$ for $x \in (a,b)$. Notice that F is continuous on $[,ab]$. In fact, we have $F(a) = \lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a+} F(x)$ and $F(b) = \lim\limits_{x\to b-} f(x) = \lim\limits_{x\to b-} F(x)$. Thus, F is continuous at $x = a$ and b. So, the function F is desired.
The last assertion is clearly follows from the continuity of F immediately. The proof is finished. $\square$

Remark 4.7 Indeed, in the proof of Proposition 4.6 $(i) \Rightarrow (ii)$ above, we have shown the following fact. Suppose that f is uniformly continuous function defined on A. If (x_n) is a Cauchy sequence in A, then so is the sequence $(f(x_n))$. We can use this simple observation to see a function "NOT" being uniformly continuous on its domain.
Notice the assumption of the uniform continuity of f is essential in here by considering the simple example that $f(x) = \frac{1}{x}$, $x \in A := (0,1]$ and $x_n = \frac{1}{n}$, $n = 1, 2....$

Definition 4.8 Let I be an interval (may be unbounded). A function s defined on I is called a step function if there exist finitely many pairwise disjoint subintervals of I, say $J_1, ..., J_N$ such that $I = \bigcup_{k=1}^{N} J_k$ and s is a constant on each J_k.

Proposition 4.9 *Let $f : (a,b) \to \mathbb{R}$ be a continuous function. Then the followings are equivalent.*

- *(i) f is uniformly continuous on (a,b).*
- *(ii) For each $\varepsilon > 0$ there exists a step function s on (a,b) such that $|f(x) - s(x)| < \varepsilon$ for all $x \in (a,b)$, that is, the function f can be "uniformly approximated" by step functions on (a,b).*

Proof: Suppose that (i) holds. Then by Proposition 4.6, there exists a continuous extension F of f on $[a,b]$. Let $\varepsilon > 0$. Then there is $\delta > 0$ so that $|F(x) - F(y)| < \varepsilon$ whenever $x, y \in (a,b)$ with $|x - y| < \delta$. Now if we choose a partition $a = x_0 < \cdots < x_n = b$ on (a,b) such that $|x_k - x_{k-1}| < \delta$ for $k = 1, ..., n$. Now if we let $s(x) := F(x_{k-1})$ when $x \in [x_{k-1}, x_k) \cap (a,b)$, then s is the step function as desired.
Now assume that (ii) holds. Let $\varepsilon > 0$. Then by the assumption, there is a step function s on (a,b) such that $|s(x) - f(x)| < \varepsilon$ for all $x \in (a,b)$. From the definition of a step function, there exist some $c, d \in (a,b)$ with $a < c < d < b$ so that $s(x) \equiv p$ on (a,c) and $s(x) \equiv q$ on (d,b) for some constants p and q. Hence, $|f(x) - p| < \varepsilon$ for any $x \in (a,c)$. Similarly, we also have $|f(x) - q| < \varepsilon$ for all $x \in (d,b)$.
It is because the restriction of f on $[c,d]$ is uniformly continuous, there is $\delta_1 > 0$ such that $|f(x) - f(x')| < \varepsilon$ for all $x, x' \in [c,d]$ with $|x - x'| < \delta_1$. On the other hand, since f is continuous at $x = c$ and d, we can find $\delta_2 > 0$ such that $|f(x) - f(c)| < \varepsilon$ as $|x - c| < \delta_2$ and $|f(x) - f(d)| < \varepsilon$ as $|x - d| < \delta_2$. Now if we take $0 < \delta < \min(\delta_1, \delta_2)$, then $|f(x) - f(x')| < 2\varepsilon$ as $x, x' \in (a,b)$ with $|x - x'| < \delta$. So, f is uniformly continuous on (a,b). The proof is finished. $\square$

In fact, in the proof of Proposition 4.9 $(i) \Rightarrow (ii)$, we have shown the following fact:

Corollary 4.10 *If f is a continuous function defined on a closed and bounded interval $[a, b]$, then it can be uniformly approximated by step functions, that is, for each $\varepsilon > 0$, there exists a step function s defined on $[a, b]$ such that $|f(x) - s(x)| < \varepsilon$ for all $x \in [a, b]$.*

References

[1] R.G. Bartle and I.D. Sherbert, Introduction to Real Analysis, (4*th ed*), Wiley, (2011).