eg13 Convert integrals between Cartesian and Polar coordinates.

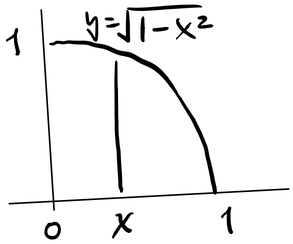
(a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta \cos\theta \, dr\, d\theta$

(b) $\int_1^2 \int_0^{\sqrt{2x-x^2}} y \, dy\, dx$

Soln (a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta \cos\theta \, dr\, d\theta$

$= \int_0^{\frac{\pi}{2}} \left[\int_0^1 r^3 \sin\theta \cos\theta \, dr \right] d\theta$

$\therefore$ Region : $0 \leq r \leq 1$, $0 \leq \theta \leq \frac{\pi}{2}$



$\Rightarrow \int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta \cos\theta \, dr\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 (r\cos\theta)(r\sin\theta) \, \underline{r\, dr\, d\theta}$

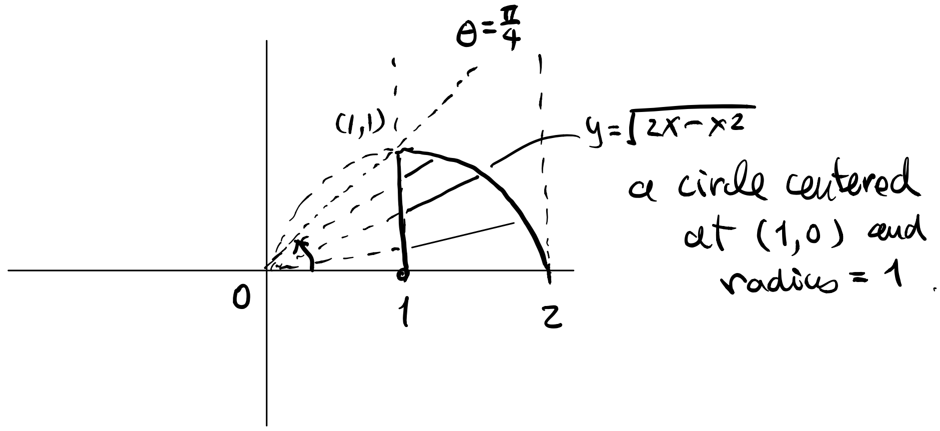
$= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} x y \, dy \right] dx$

or simply $= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy\, dx$.

$\left(\text{or } \int_0^1 \int_0^{\sqrt{1-y^2}} xy \, dx\, dy \right)$

(b) $\int_1^2 \int_0^{\sqrt{2x-x^2}} y \, dy\, dx = \int_1^2 \left[\int_0^{\sqrt{2x-x^2}} y \, dy \right] dx$

$\therefore$ region is : $1 \le x \le 2$, $0 \le y \le \sqrt{2x-x^2}$



The curve $x = 1$

$\Leftrightarrow r\cos\theta = 1 \Leftrightarrow r = \frac{1}{\cos\theta} = \sec\theta \quad (0 \le \theta \le \frac{\pi}{4})$

The curve $y = \sqrt{2x-x^2}$

$\Leftrightarrow r\sin\theta = \sqrt{2r\cos\theta - r^2\cos^2\theta}$

$\Leftrightarrow r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$

$\Leftrightarrow r^2 = 2r\cos\theta \quad$ (since $r > 0$)

$\Leftrightarrow r = 2\cos\theta \quad (0 \le \theta \le \frac{\pi}{2})$

Hence $\int_1^2 \int_0^{\sqrt{2x-x^2}} y\,dy\,dx$

$$= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} y\,\underbrace{dy\,dx}$$

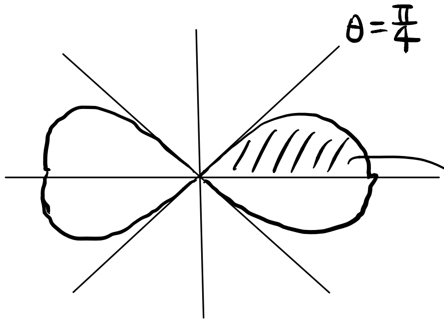
$$= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} (r\sin\theta)(r\,dr\,d\theta)$$

$$= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} r^2\sin\theta\,dr\,d\theta .$$

#

eg 14: Find area enclosed by $r^2 = 4\cos 2\theta$

Soln



(Symmetric both in x & y directions)

$\frac{1}{4}$ of the area.

> **Remark**: r is "not really" a function of θ, it should be regarded as a "level set":
>
> (i) there is no soln. when $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ & $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$.
>
> (ii) in terms of (x,y) coordinates
>
> $$F(x,y) = (x^2+y^2)^2 - 4(x^2-y^2) = 0$$
>
> which has a critical point at $(x,y) = (0,0)$ on the level set (Implicit Function Theorem).

By the symmetry

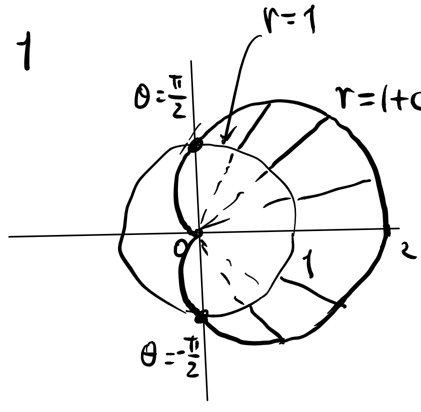
$$\text{Area} = 4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{4\cos 2\theta}} 1 \cdot r\,dr\,d\theta = 8\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta = 4 \quad (\text{check!})$$

#

eg 15: Integrate $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ over the region R bounded between

$$\begin{cases} r = 1 + \cos\theta & (\text{cardioid}) \\ r = 1 & (\text{circle}) \end{cases}$$

and outside the circle $r = 1$



Soln: Intersections:

$$1+\cos\theta = 1$$

$$\Leftrightarrow \quad \cos\theta = 0$$

$$\Leftrightarrow \quad \theta = \frac{\pi}{2} + k\pi$$

In practice, we can choice $\theta = \pm \frac{\pi}{2}$.

$$\therefore \iint_R f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{1+\cos\theta} \frac{1}{r}\, r\,dr\,d\theta = 2 \quad \text{(check!)}$$

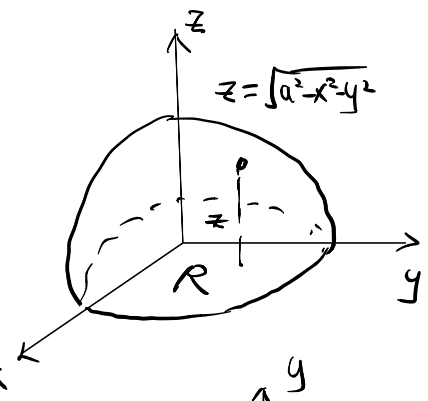
##

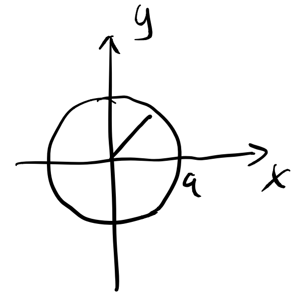
eg16 Let $z = \sqrt{a^2 - x^2 - y^2}$ be a function defined on

$$R = \{(x,y) : x^2 + y^2 \le a^2\}$$

(The graph of z is the hemisphere of radius a)

Find the <u>average</u> <u>height</u> of the hemisphere.





<u>Soln</u>: Average height $= \frac{1}{\text{Area}(R)} \iint_R z\,dA$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}\; r\,dr\,d\theta$$

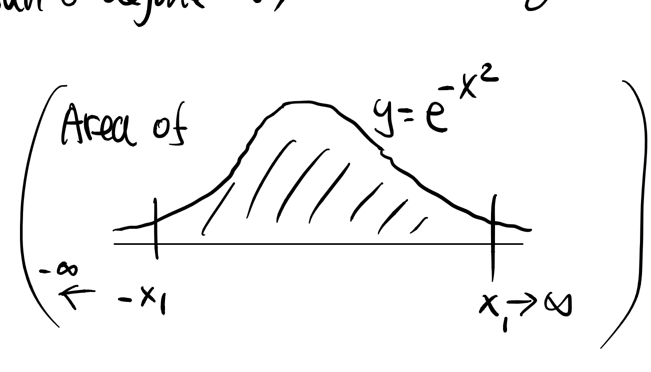
$$= \frac{1}{\pi a^2} \cdot \frac{2\pi a^3}{3} \quad \text{(check!)}$$

$$= \frac{2a}{3}.$$

##

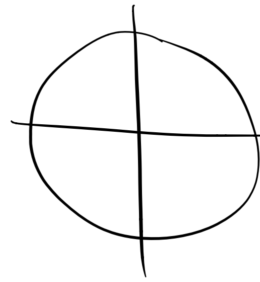
eg17 (<u>Improper integral</u>, we didn't define it, but it is a good practice example)

Find $\int_{-\infty}^{\infty} e^{-x^2}\,dx$



<u>Soln</u>: Consider $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA$

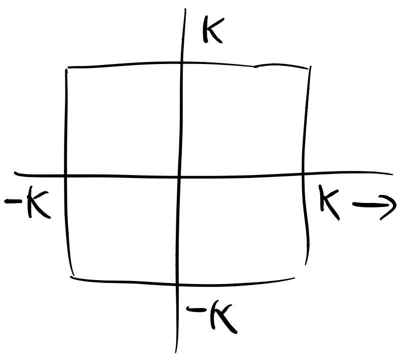
$$= \lim_{\rho \to \infty} \iint_{\{x^2+y^2 \le \rho^2\}} e^{-(x^2+y^2)} dA$$



$$= \lim_{\rho \to \infty} \int_0^{2\pi} \int_0^{\rho} e^{-r^2}\, r\,dr\,d\theta$$

$$= \lim_{\rho \to \infty} \frac{1}{2} \int_0^{2\pi} (1 - e^{-\rho^2})\, d\theta$$

$$= \lim_{\rho \to \infty} \pi (1 - e^{-\rho^2}) = \pi$$



On the other hand

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA$$

$$= \lim_{K \to +\infty} \int_{-K}^{K} \int_{-K}^{K} e^{-x^2-y^2}\, dx\,dy$$

$$= \lim_{K \to +\infty} \int_{-K}^{K} \int_{-K}^{K} e^{-x^2} e^{-y^2}\, dx\,dy$$

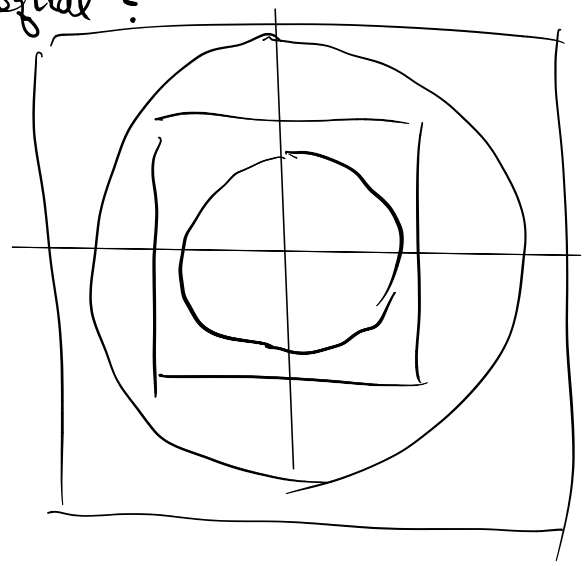
$$= \lim_{K \to +\infty} \int_{-K}^{K} \left[e^{-y^2} \int_{-K}^{K} e^{-x^2} dx \right] dy$$

$$= \lim_{K \to +\infty} \left(\int_{-K}^{K} e^{-x^2} dx \right) \left(\int_{-K}^{K} e^{-y^2} dy \right)$$

$$= \lim_{K \to +\infty} \left(\int_{-K}^{K} e^{-x^2} dx \right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \quad \#$$

[Caution: we are calculating $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA$ using two different limiting processes. Why are they equal?]



Answer:

Triple Integrals

Def 5 Let $f(x,y,z)$ be a function defined on a (closed and bounded) rectangular box $B=[a,b]\times[c,d]\times[r,s]$

Then the triple integral of f over the box B is

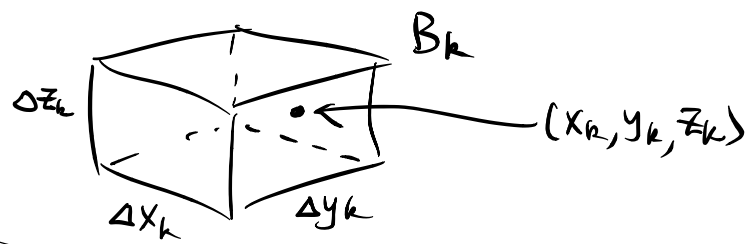
$$\iiint_B f(x,y,z)\,dV = \lim_{\|P\|\to 0} \sum_k f(x_k,y_k,z_k)\,\Delta V_k$$

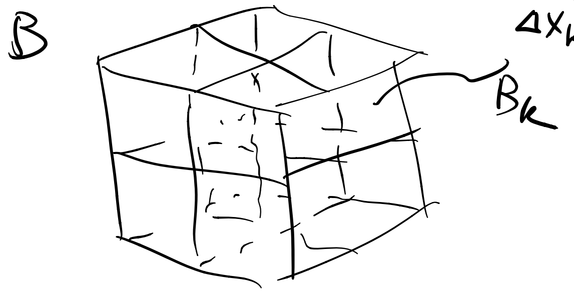
if this exists.

where (i) $P=P_1\times P_2\times P_3$ is a subdivision of B into sub-rectangular boxes by partitions P_1, P_2, P_3 of $[a,b]$, $[c,d]$, $[r,s]$ respectively, and

$$\|P\| = \max(\|P_1\|, \|P_2\|, \|P_3\|)$$

(ii) (x_k,y_k,z_k) is an arbitrary point in a sub-rectangular box B_k





(iii) $\Delta V_k = \mathrm{Vol}(B_k) = \Delta x_k\,\Delta y_k\,\Delta z_k$.

Thm 4 (Fubini's Theorem for Triple Integrals (1st form))

If $f(x,y,z)$ is continuous (in fact, integrable is sufficient) on $B=[a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Note: Interchanging the order of the coordinates, we also have

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$$

$$= \cdots \text{ in any order of } dx, dy, dz.$$

Def 6 (Triple integral over a general region $D \subset \mathbb{R}^3$)

Let $f(x,y,z)$ be a function on a closed and bounded region $D \subset \mathbb{R}^3$. Then

$$\iiint_D f(x,y,z)\,dV \stackrel{\text{def}}{=\!=} \iiint_B F(x,y,z)\,dV$$

where B is a closed and bounded rectangular box containing D, and

$$F(x,y,z) = \begin{cases} f(x,y,z), & \text{if } (x,y,z) \in D \\ 0, & \text{if } (x,y,z) \in B \setminus D. \end{cases}$$

Note: As in double integral, this definition is well-defined.

Special types of closed and bounded region $D \subset \mathbb{R}^3$

(1) $D = \{(x,y,z) : (x,y) \in R_1,\ u_1(x,y) \leq z \leq u_2(x,y)\}$

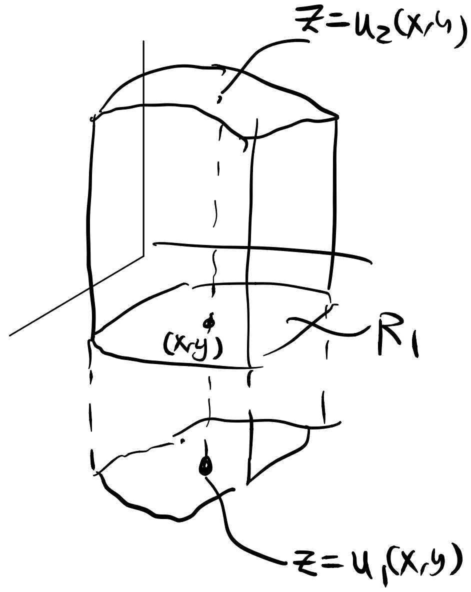
$(u_1(x,y) \leq u_2(x,y),\ u_1 \not\equiv u_2)$

(2) $D = \left\{ (x,y,z) : \begin{array}{l} (x,z) \in R_2 \\ v_1(x,z) \leq y \leq v_2(x,z) \end{array} \right\}$,

$(v_1 \leq v_2,\ v_1 \not\equiv v_2)$

(3) $D = \left\{ (x,y,z) : \begin{array}{l} (y,z) \in R_3 \\ w_1(y,z) \leq x \leq w_2(y,z) \end{array} \right\}$

$(w_1 \leq w_2,\ w_1 \not\equiv w_2)$



where R_i, $i = 1,2,3$, are closed and bounded plane regions and u_1, u_2; v_1, v_2; w_1, w_2 are continuous wrt the corresponding variables.

Thm 5 (Fubini's Thm for triple integrals (strong form))

Let $f(x,y,z)$ be a continuous (integrable) function on D.

If D is of type (1) as above, then

$$\iiint_D f(x,y,z)\,dV = \iint_{R_1} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dx\,dy$$

Similarly for types (2) & (3)

Note: Particular, we have (using Fubini's for double integrals):

if $D = \left\{ (x,y,z) : \begin{array}{l} a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x) \\ u_1(x,y) \leq z \leq u_2(x,y) \end{array} \right\}$

(i.e. R_1 is of type (1)), then

$$\iiint_D f(x,y,z)\,dV = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dy \right] dx$$

or simply

$$\boxed{\iiint_D f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

Similarly for other types

Prop 6: The propositions 1-4 for double integrals also hold for triple integrals over closed and bounded region in $\mathbb{R}^3$