MATH 2020 HW3 sol

15.7 : 10, 13, 17, 34, 37, 47, 52, 60, 66, 83.

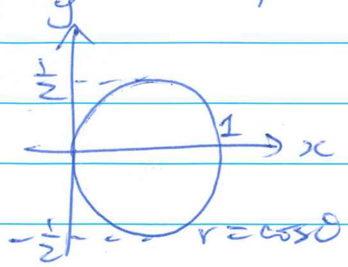
(15.7.10). $\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r\sin\theta + 1)\, r \, d\theta\, dz\, dr$

$= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} (0 + 2\pi r)\, dz\, dr = 2\pi \int_0^2 r\left[\sqrt{4-r^2} - (r-2)\right] dr$

$\cancel{= \int_0^2 \pi\left[(4-r^2) - (r-2)^2\right] dr}$

$= 8\pi$

(15.7.13) $\int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} \int_0^{3r^2} f(r,\theta,z)\, dz\, r\, dr\, d\theta$



(15.7.17) $\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} \int_0^4 f(r,\theta,z)\, dz\, r\, dr\, d\theta$

(15.7.34) required volume

$= \int_0^{\pi/2} \int_1^{1+\cos\varphi} \int_0^{2\pi} \rho^2 \sin\varphi \, d\theta\, d\rho\, d\varphi$

$= 2\pi \int_0^{\pi/2} \int_1^{1+\cos\varphi} \rho^2 \sin\varphi \, d\rho\, d\varphi$

$= 2\pi \int_0^{\pi/2} \frac{(1+\cos\varphi)^3 - 1}{3} \sin\varphi \, d\varphi$

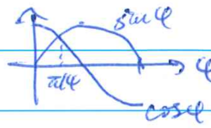
$= \frac{11}{6}\pi$

(15.7.37). Consider the domain of integration.

For $(x,y,z) = (\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)$,

$\rho \geq 0,\ 0 \leq \theta < 2\pi,\ 0 \leq \varphi \leq \pi,$

$$z \leq \sqrt{x^2+y^2} \iff \rho\cos\varphi \leq \rho\sin\varphi$$
$$\iff \rho = 0 \text{ or } \tfrac{\pi}{4} \leq \varphi \leq \pi$$



Note that the region defined by $z \leq \sqrt{x^2+y^2}$ lies below the horizontal plane passing through the centre of the sphere defined by $\rho = 2\cos\varphi \iff x^2+y^2+(z-1)^2 = 1$, hence the domain of integration lies inside the sphere, ie it is a subset of the region defined by $x^2+y^2+(z-1)^2 \leq 1$
$\iff \rho \leq 2\cos\varphi$.

$\therefore$ The vol

$$= \int_{\pi/4}^{\pi/2}\int_0^{2\cos\varphi}\int_0^{2\pi} \rho^2\sin\varphi \, d\theta\, d\rho\, d\varphi$$

$$= 2\pi \int_{\pi/4}^{\pi/2}\int_0^{2\cos\varphi} \rho^2 \sin\varphi \, d\rho\, d\varphi$$

$$= 2\pi \int_{\pi/4}^{\pi/2} \frac{(2\cos\varphi)^3}{3}\sin\varphi \, d\varphi$$

$$= \frac{\pi}{3}$$

(15.7.47) required vol

$$= \int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{\sqrt{1-r^2}} r \, dz\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{\sin\theta} r\sqrt{1-r^2}\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}\left(1 - (1-\sin^2\theta)^{3/2}\right) d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}(1-\cos^3\theta)\, d\theta$$

$$= \frac{3\pi-4}{18}$$

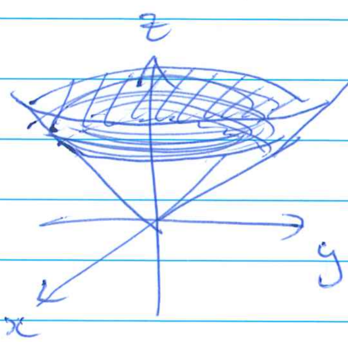
52). required volume

$$= \int_1^2 \int_0^z \int_0^{2\pi} r \, d\theta \, dr \, dz$$

$$= \int_1^2 \int_0^z 2\pi r \, dr \, dz$$

$$= \int_1^2 \pi z^2 \, dz$$

$$= \frac{7}{3}\pi$$



60) required volume

$$= \int_0^{2\pi} \int_1^3 \int_0^{9-r^2} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_1^3 (9-r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi} 16 \, d\theta$$

$$= 32\pi$$

66) required average

$$= \frac{1}{\frac{1}{2}(\frac{4}{3}\pi)} \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} (\rho \cos\varphi)(\rho^2 \sin\varphi) \, d\theta \, d\varphi \, d\rho$$

$$= 3 \int_0^1 \int_0^{\pi/2} \rho^3 \sin\varphi \cos\varphi \, d\varphi \, d\rho$$

$$= 3 \int_0^1 \rho^3 \left[\frac{\sin^2 \pi/2}{2}\right] d\rho$$

$$= \frac{3}{2} \int_0^1 \rho^3 \, d\rho$$

$$= \frac{3}{8}.$$

83 a) A plane perpendicular to the x-axis is implicitly defined by $x = a$.

$\Leftrightarrow r\cos\theta = a$

$\Leftrightarrow r = a\sec\theta$

b) A plane perpendicular to the y-axis is implicitly defined by $y = b$

$\Leftrightarrow r\sin\theta = b$

$\Leftrightarrow r = b\csc\theta$