MATH 2020 Advanced Calculus II

Tutorial 8

1. Compute $\displaystyle\int_C \sqrt{x^2+y^2}ds$ where C is parametrized by $\gamma(t) = (\cos t, \sin t, t),\ t \in [-2\pi, 2\pi]$.

   Solution. We have

$$ds = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt = \sqrt{2}\, dt$$

   and so

$$\begin{aligned}\int_C \sqrt{x^2+y^2}\, ds &= \int_{-2\pi}^{2\pi} \sqrt{(\cos t)^2 + (\sin t)^2} \cdot \sqrt{2}\, dt \\ &= \sqrt{2}\,[2\pi - (-2\pi)] \\ &= 4\sqrt{2}\pi\end{aligned}$$

2. Compute $\displaystyle\int_C F \cdot dr$ where $F(x,y,z) = (x,y,z)$ and C is the broken curve from $(0,0,0)$ to $(1,1,1)$ which is the union of $\{(t,t^2,0)|\ t \in [0,1]\}$ and $\{(1,1,t)|\ t \in [0,1]\}$.

   Solution.

$$\begin{aligned}\int_C F \cdot dr &= \int_0^1 (t,t^2,0)\cdot(1,2t,0)\, dt + \int_0^1 (1,1,t)\cdot(0,0,1)\, dt \\ &= \int_0^1 (t+2t^3)dt + \int_0^1 tdt \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\ &= \frac{3}{2}\end{aligned}$$

3. Compute $\displaystyle\int_C F{\cdot}n\, ds$ where $F(x,y) = \left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right)$ and C is parametrized by $(\cos t, 2\sin t),\ t \in [0,2\pi]$.

$$\begin{aligned}\int_C F \cdot n\, ds &= \int_C F_1 dy - F_2 dx \\ &= \int_0^{2\pi} \left[\frac{-2\sin t}{\sqrt{\cos^2 t + 4\sin^2 t}} \cdot 2\cos t - \frac{\cos t}{\sqrt{\cos^2 t + 4\sin^2 t}} \cdot (-\sin t)\right] dt \\ &= -3\int_0^{2\pi} \frac{\sin t\cos t}{\sqrt{\cos^2 t + 4\sin^2 t}}\, dt \\ &= 0\end{aligned}$$