Proof of Divergence Theorem

Same as Green's Thm, we'll prove only the case of special domain D which is of type I, II and III:

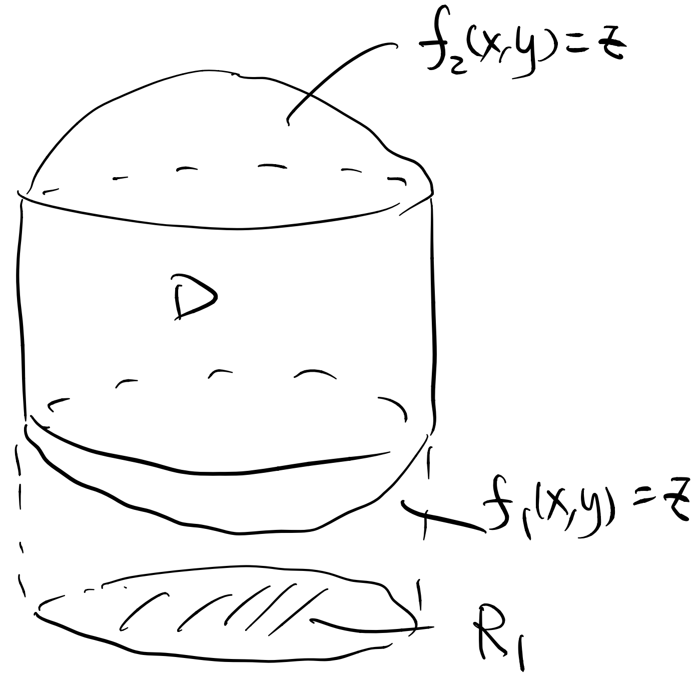
i.e.

$$D = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R_1,\ f_1(x,y) \leq z \leq f_2(x,y)\} \quad (\text{type I})$$

$$= \{(x,y,z) \in \mathbb{R}^3 : (y,z) \in R_2,\ g_1(y,z) \leq x \leq g_2(y,z)\} \quad (\text{type II})$$

$$= \{(x,y,z) \in \mathbb{R}^3 : (z,x) \in R_3,\ h_1(z,x) \leq y \leq h_2(z,x)\} \quad (\text{type III})$$

eg of type I domain :



And also as in the proof of Green's Theorem,

for $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$,

we'll prove 3 equalities in the following which combine to give the divergence thm:

$$\begin{cases} \displaystyle\iint_S M\hat{i}\cdot\vec{n}\,d\sigma = \iiint_D \frac{\partial M}{\partial x}\,dV & \text{(by type II)} \\ \displaystyle\iint_S N\hat{j}\cdot\vec{n}\,d\sigma = \iiint_D \frac{\partial N}{\partial y}\,dV & \text{(by type III)} \\ \displaystyle\iint_S L\hat{k}\cdot\vec{n}\,d\sigma = \iiint_D \frac{\partial L}{\partial z}\,dV & \text{(by type I)} \end{cases}$$

The proof are similar, we'll prove only the last one:

$$\iint_S L\hat{k}\cdot\vec{n}\,d\sigma = \iiint_D \frac{\partial L}{\partial z}\,dV.$$

By Fubini's Thm,

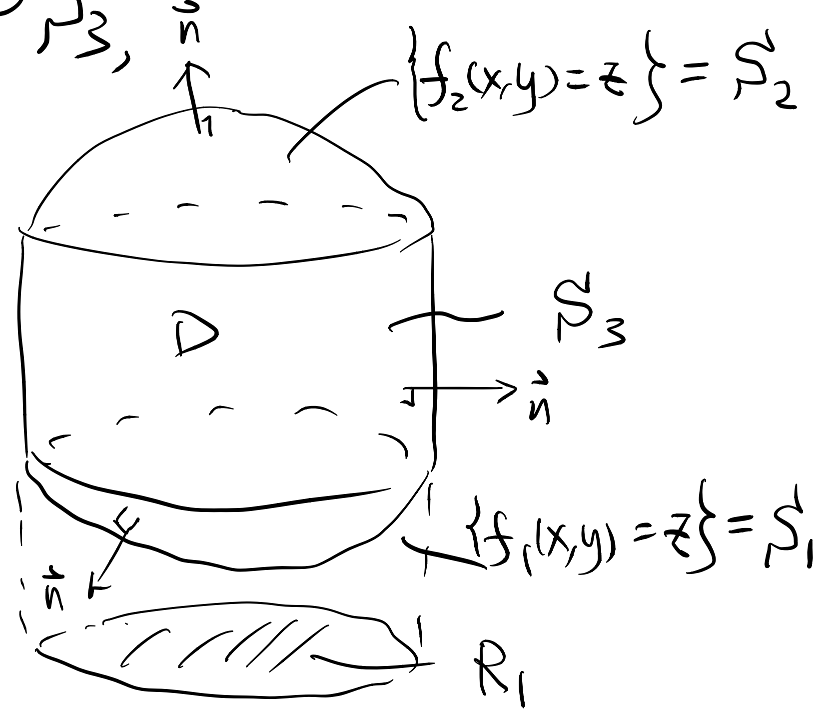
$$\text{R.H.S.} = \iiint_D \frac{\partial L}{\partial z}\,dV = \iint_{R_1}\left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial L}{\partial z}\,dz\right]dx\,dy$$

$$= \iint_{R_1}\left[L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))\right]dx\,dy$$

For the L.H.S., we note by definition of type I domain, the boundary S of D can be written

as $\quad S = S_1 \cup S_2 \cup S_3,$

where



S_1 = graph of f_1

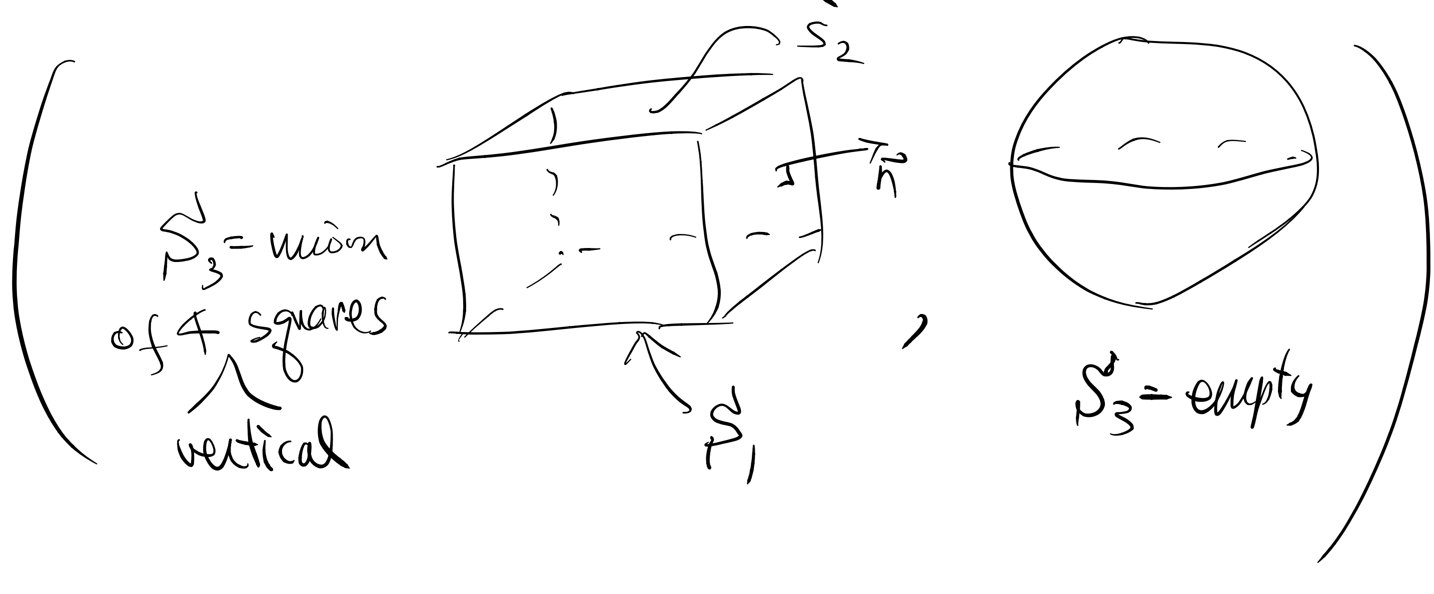
$= \{(x, y, f_1(x,y))\}$

$= \{ z = f_1(x,y) \}$,

S_2 = graph of f_2

$= \{(x, y, f_2(x,y))\} = \{ z = f_2(x,y) \}$, and

S_3 = a vertical surface (which could be <u>empty</u>!)



Hence

$$L.H.S. = \iint_{\tilde{S}} L\,\hat{k}\cdot\vec{n}\,d\sigma = \iint_{\tilde{S}_1} L\,\hat{k}\cdot\vec{n}\,d\sigma + \iint_{\tilde{S}_2} L\,\hat{k}\cdot\vec{n}\,d\sigma$$

$$+ \iint_{\tilde{S}_3} L\,\cancelto{0}{\hat{k}\cdot\vec{n}}\,d\sigma$$

(Since $\vec{n}$ of vertical surface is horizontal!)

Now on the upper surface $\tilde{S}_2 = \{z = f_2(x,y)\}$, the outward normal $\vec{n}$ is upward (in the sense that $\vec{n}\cdot\hat{k} > 0$). Note that the parametrization

$$(x,y) \longmapsto \vec{r}(x,y) = x\hat{i} + y\hat{j} + f_2(x,y)\hat{k}$$

has

$$\begin{cases} \vec{r}_x = \hat{i} + \dfrac{\partial f_2}{\partial x}\hat{k} \\[2mm] \vec{r}_y = \hat{j} + \dfrac{\partial f_2}{\partial y}\hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \dfrac{\partial f_2}{\partial x} \\ 0 & 1 & \dfrac{\partial f_2}{\partial y} \end{vmatrix} = -\frac{\partial f_2}{\partial x}\hat{i} - \frac{\partial f_2}{\partial y}\hat{j} + \hat{k}$$

+ve $\Rightarrow$ $\vec{r}_x \times \vec{r}_y$ is upward

$\therefore \quad \vec{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$

and $\hat{k} \cdot \vec{n} = \dfrac{1}{|\vec{r}_x \times \vec{r}_y|}$

Hence

$$\iint_{S_2} L \hat{k} \cdot \vec{n} \, d\sigma = \iint_{R_1} L(x, y, f_2(x,y)) \underbrace{\frac{1}{|\vec{r}_x \times \vec{r}_y|}}_{\hat{k} \cdot \vec{n}} \cdot \underbrace{|\vec{r}_x \times \vec{r}_y| \, dA}_{d\sigma}$$

$$= \iint_{R_1} L(x, y, f_2(x, y)) \, dx \, dy \, .$$

Similarly, note that the outward normal at S_1 (lower surface) is <u>downward</u>, (i.e. $\vec{n} \cdot \hat{k} < 0$)

we have $\vec{n} = - \dfrac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$, where $\vec{r} = x\hat{i} + y\hat{j} + f_1(x,y)\vec{k}$

& $\hat{k} \cdot \vec{n} = - \dfrac{1}{|\vec{r}_x \times \vec{r}_y|}$. (Check!)

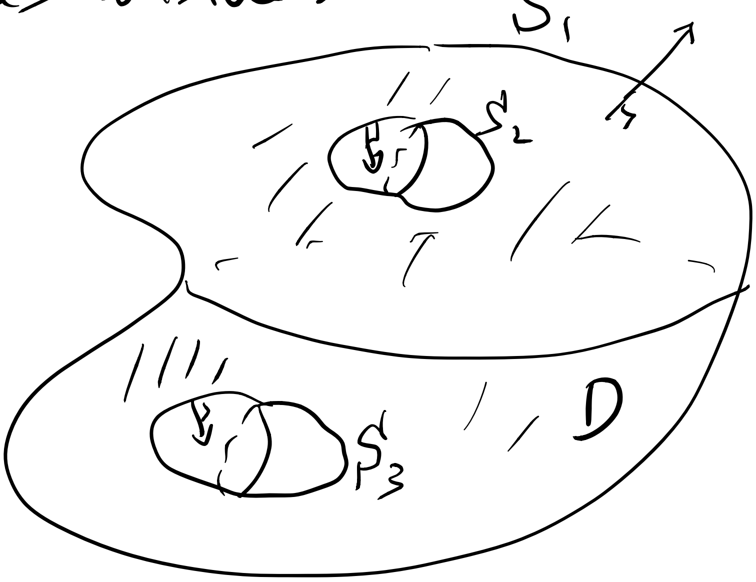
Hence $\displaystyle\iint_{S_1} L \hat{k} \cdot \vec{n} \, d\sigma = - \iint_{R_1} L(x, y, f_1(x,y)) \, dx dy$

Therefore,

$$\iint_S L\vec{k}\cdot\vec{n}\,d\sigma = \iint_{R_1}\left[L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))\right]dxdy$$

$$= \iiint_D \frac{\partial L}{\partial z}\,dV.$$

Note: Similar to Green's Theorem, the Divergence Thm is also hold for solid region with finitely many holes inside:



$$\iiint_D \vec{\nabla}\cdot\vec{F}\,dV = \sum_{i=1}^{k}\iint_{S_i}\vec{F}\cdot\vec{n}\,d\sigma$$

for $\vec{n}$ outward normal with respect to D.

<u>Note</u> : Physical meaning of $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$ in $\mathbb{R}^3$

$=$ <u>flux density</u> (by the divergence theorem)

Unified treatment of Green's, Stokes', and Divergence Theorems:

Stokes' Theorem in notations of differential forms (in $\mathbb{R}^3$)

Working definition of differential forms:

(1) A differential 1-form (or simply 1-form) is a linear combination of the symbols dx, dy, and dz:

$$\boxed{\omega = \omega_1\, dx + \omega_2\, dy + \omega_3\, dz}$$

with coefficients $\omega_1, \omega_2, \omega_3$ are functions on $\mathbb{R}^3$.

eg: The total differential of a smooth function f is a differential 1-form:

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz .$$

(2) <u>Wedge production</u>: Let "$\wedge$" be an operation such that

$$\begin{cases} dx \wedge dx = dy \wedge dy = dz \wedge dz = 0 \\ dx \wedge dy = -dy \wedge dx \\ dy \wedge dz = -dz \wedge dy \\ dz \wedge dx = -dx \wedge dz \end{cases}$$

and usual rules in arithmetic:

i.e. If $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$
$\eta = \eta_1 dx + \eta_2 dy + \eta_3 dz$,

then

$$\begin{aligned} \omega \wedge \eta &= (\omega_1 dx + \omega_2 dy + \omega_3 dz) \wedge (\eta_1 dx + \eta_2 dy + \eta_3 dz) \\ &= \cancel{\omega_1 dx \wedge \eta_1 dx} + \omega_2 dy \wedge \eta_1 dx + \omega_3 dz \wedge \eta_1 dx \\ &+ \omega_1 dx \wedge \eta_2 dy + \cancel{\omega_2 dy \wedge \eta_2 dy} + \omega_3 dz \wedge \eta_2 dy \\ &+ \omega_1 dx \wedge \eta_3 dz + \omega_2 dy \wedge \eta_3 dz + \cancel{\omega_3 dz \wedge \eta_3 dz} \\ &= (\omega_1 \eta_2 - \omega_2 \eta_1)\, dx \wedge dy \\ &\quad + (\omega_2 \eta_3 - \omega_3 \eta_2)\, dy \wedge dz \\ &\quad\quad + (\omega_3 \eta_1 - \omega_1 \eta_3)\, dz \wedge dx \end{aligned}$$

$$\therefore \boxed{\begin{aligned} \omega \wedge \eta = &(\omega_2\eta_3 - \omega_3\eta_2)\, dy \wedge dz \\ &+ (\omega_3\eta_1 - \omega_1\eta_3)\, dz \wedge dx \\ &+ (\omega_1\eta_2 - \omega_2\eta_1)\, dx \wedge dy \end{aligned}}$$

- Linear combinations of $dy \wedge dz$, $dz \wedge dx$, & $dx \wedge dy$ are called <u>differential 2-forms</u> (on $\mathbb{R}^3$):

$$\boxed{\zeta = \zeta_1\, dy \wedge dz + \zeta_2\, dz \wedge dx + \zeta_3\, dx \wedge dy}$$