MATH1010 I

w3-1

Foreword: A nice video & lecture note can be found at: socratic.org

After entering the webpage, look for Calculus under "Subjects".

There, look for "Derivatives". Then go to "Differentiable vs. Non-differentiable Functions".

Now you should be able to find some nice accompanying videos, e.g. the following one.

https://www.youtube.com/watch?v=I7nK7zSbLg4&index=17&list=PL265CB737C01F8961

Introduction. Today, we talked about the two derivatives below:

$$(1)\ \frac{d \sin x}{dx} = \cos x , \qquad (2)\ \frac{d\, e^x}{dx} = e^x$$

The proofs of both of them use:

- (Periodicity of the sine function, of the exponential function). More precisely, we used (first of all), the formulas $\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$ and $e^{x+h} = e^x e^h$.
- Then we used for (1), the limit formula $\lim_{h\to 0} \frac{\sin(h)}{h} = 0$. (This is usually proved via a picture, see e.g. http://www.ies-math.com/math/java/calc/LimSinX/LimSinX.html
- Remark. But a more "rigorous" way is to use the following "complicated" definition of sine function, i.e. $\sin h = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \cdots$ (Physics textbook would have something like $\sin(h) \approx h$, when h small.)
- For (2), we use the limit formula $\lim_{h\to} \frac{e^h - 1}{h} = 0$. (This will follow from (i) $e^h = 1 + h + \frac{h^2}{2!} + \cdots$ and (ii) the Sandwich/Squeeze Theorem (More about this later!))

Proofs

For (1).

Step 1) Consider the fraction $\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$

$$= \frac{\sin(x)\,(\cos(h) - 1)}{h} + \frac{\sin(h)}{h}\cos(x)$$

Step 2) We compute the limits $\lim_{h \to 0} \frac{\cos(h)-1}{h}$ by "relating" (cosine) to (sine).

That is, use the "double angle formula" (i.e. $\cos(2h) = 1 - \sin^2(h)$, or $\cos(h) = 1 - \sin^2\left(\frac{h}{2}\right)$) and obtain $\cos(h) - 1 = -\sin^2\left(\frac{h}{2}\right)$.

Dividing this by h, we get $\frac{\cos(h)-1}{h} = \frac{-\sin^2\left(\frac{h}{2}\right)}{h} = -\frac{\sin^2\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)2} = -\frac{\left(\frac{h}{2}\right)\sin^2\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)2\left(\frac{h}{2}\right)}$

$$= -\frac{\left(\frac{h}{2}\right)\sin^2\left(\frac{h}{2}\right)}{2\left(\frac{h}{2}\right)^2} = -\left(\frac{h}{2}\right)\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)^2$$

Step 3) Now we take limit, i.e. we let $h \to 0$ in the above fraction. Then we get

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = \lim_{h \to 0} -\left(\frac{h}{2}\right)\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)^2$$

$$= \lim_{h \to 0} -\left(\frac{h}{2}\right) \lim_{h \to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)^2$$

Now when $h \to 0$, we also have $\frac{h}{2} \to 0$, so $\lim_{h \to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)^2 = \lim_{\frac{h}{2} \to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)^2 = 1.$

Combining all these, we obtain $\lim_{h \to 0} \frac{\cos(h)-1}{h} = = \lim_{h \to 0} -\left(\frac{h}{2}\right) \lim_{h \to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)^2 = 0 \times 1 = 0.$

Step 4) We still have one more term to handle in Step 1), i.e. the term $\frac{\sin(h)}{h}\cos(x)$.

This is straightforward. $\lim_{h \to 0} \frac{\sin(h)}{h} \cdot \cos x = \cos x \cdot \lim_{h \to 0} \frac{\sin(h)}{h} = \cos x \cdot 1 = \cos x$

Step 5). Combining all the above, we get $\frac{d \sin(x)}{dx} = \lim_{h \to 0} \frac{\sin(x+h)-\sin(x)}{h} = \cos(x).$

For (2).

The idea is similar.

Consider the fraction $\frac{e^{x+h}-e^x}{h} = \frac{e^x e^h - e^x}{h} = e^x \left(\frac{e^h-1}{h}\right).$

Now use the formula $e^h = 1 + h + \frac{h^2}{2!} + \cdots$ to obtain

$$e^h - 1 = h + \frac{h^2}{2!} + \cdots$$

Dividing again by h, we get $\frac{e^h-1}{h} = \frac{h+\frac{h^2}{2!}+\cdots}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots$

Now we can use the Squeeze/Sandwich Theorem to argue that $\lim_{h\to 0} \frac{h}{2!} + \frac{h^2}{3!} + \cdots = 0$. (Please think about this! I will explain it later).

Conclusion: We obtain $\lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0} \frac{e^h-1}{h} = e^x$.

Why the two proofs are related?

This is due to the famous Euler's formula, which says, if we denote $\sqrt{-1}$ by the symbol i, then we get

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{2!} + \frac{(ix)^4}{2!} + \cdots \\
&= 1 + ix + \frac{iix^2}{2!} + \frac{iiix^3}{2!} + \frac{iiiix^4}{2!} + \cdots \\
&= 1 + ix + \frac{(-1)x^2}{2!} + \frac{(-1)ix^3}{2!} + \frac{(-1)(-1)x^4}{2!} + \cdots \\
&= 1 + ix + \frac{(-1)x^2}{2!} + \frac{(-i)x^3}{2!} + \frac{x^4}{2!} + \cdots \\
&= 1 + \frac{(-1)x^2}{2!} + \frac{(-1)(-1)x^4}{2!} + \cdots + ix + \frac{(-1)ix^3}{2!} + \cdots \\
&= 1 + \frac{(-1)x^2}{2!} + \frac{(-1)(-1)x^4}{2!} + \cdots + i\left(x + \frac{(-1)x^3}{2!} + \cdots\right) \\
&= \cos(x) + i\sin(x)
\end{aligned}$$

because one can define (this is the modern definition) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$