THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5510 Foundation of Advanced Mathematics 2017-2018
Supplementary Exercise 2

1. Let $S = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}$.

   Define a relation $\sim$ on S such that $f \sim g$ if and only if $f'(x) = g'(x)$ for all $x \in \mathbb{R}$.

   (a) Show that the relation $\sim$ is an equivalence relation.

   (b) Let $f \in S$, what are the elements of the equivalence class $[f]$?

   Ans:

   (a) i. (Reflexive) $f \sim f$ as $f'(x) = f'(x)$ for all $x \in \mathbb{R}$.

   ii. (Symmetric) If $f \sim g$, then $f'(x) = g'(x)$ which is just $g'(x) = f'(x)$ for all $x \in \mathbb{R}$, so $g \sim f$.

   iii. (Transitive) If $f \sim g$ and $g \sim h$, then $f'(x) = g'(x)$ and $g'(x) = h'(x)$ for all $x \in \mathbb{R}$, so $f'(x) = h'(x)$ for all $x \in \mathbb{R}$ and $f \sim h$.

   Therefore, $\sim$ is an equivalence relation.

   (b) $f \sim g$ if and only if $f'(x) = g'(x)$, i.e $f'(x) - g'(x) = 0$ for all $x \in \mathbb{R}$.

   Therefore, $f \sim g$ if and only if $g(x) = f(x) + C$ for some constant C, and

$$[f] = \{g \in S : f \sim g\} = \{f + C : C \in \mathbb{R}\}.$$

2. Define an equivalence relation $\sim$ on $\mathbb{Z}$ such that $a \sim b$ if and only if $b - a$ is divisible by 5.

   (a) Show that the multiplication on $\mathbb{Z}$ induces a multiplication on $\mathbb{Z}_5 = \mathbb{Z}/\sim$.

   (b) Show that the induced multiplication on $\mathbb{Z}_5$ is commutative.

   Ans:

   (a) Let $m, m', n, n' \in \mathbb{Z}_5$ such that $m \sim m'$ and $n \sim n'$.

   Then $m' - m = 5M$ and $n' - n = 5N$ for some integers M and N.

   $m'n' - mn = (5M + m)(5N + n) - mn = 5(5MN + Mn + mN)$ where $5MN + Mn + mN$ is an integer, so $mn \sim m'n'$. Therefore, multiplication on $\mathbb{Z}$ induces a multiplication on $\mathbb{Z}_5$.

   (b) Let $[m], [n] \in \mathbb{Z}_5$, where $m, n \in \mathbb{Z}$. Then

$$[m] \cdot [n] = [m \cdot n] = [n \cdot m] = [n] \cdot [m]$$

   Therefore, the induced multiplication on $\mathbb{Z}_5$ is commutative.

3. Let $\mathbb{R}[x]$ be the set of all polynomials with real coefficients.

   Define a relation $\sim$ on $\mathbb{R}[x]$ such that $P(x) \sim Q(x)$ if and only if $Q(x) - P(x)$ is divisible by $x^2 + 1$.

   (a) Show that the relation $\sim$ is an equivalence relation.

(b) Show that for any polynomial $P(x)$, there exists $ax+b \in \mathbb{R}[x]$ such that $[P(x)] = [ax+b]$, i.e. the equivalence class of $P(x)$ is the same as the equivalence class for some linear polynomial $ax+b$.

(c) Let $ax+b, cx+d \in \mathbb{R}[x]$. Show that $[ax+b] = [cx+d]$ if and only if $a = c$ and $b = d$.

(d) What is $\mathbb{R}[x]/\sim$?

(e) Show that the multiplication on $\mathbb{R}[x]$ induces an multiplication on $\mathbb{R}[x]/\sim$.

(f) What is $[2x+3] \cdot [3x+1]$?

Ans:

(a) i. (Reflexive) $P(x) \sim P(x)$ as $P(x) - P(x) = 0$ which is divisible by x^2+1 for all $P(x) \in \mathbb{R}[x]$.

ii. (Symmetric) If $P(x) \sim Q(x)$, then $Q(x) - P(x)$ is divisible by x^2+1 and so $P(x) - Q(x) = -(Q(x) - P(x))$ is also divisible by x^2+1 which means $Q(x) \sim P(x)$.

iii. (Transitive) If $P(x) \sim Q(x)$ and $Q(x) \sim R(x)$, then $Q(x) - P(x)$ and $R(x) - Q(x)$ is divisible by x^2+1 and so $R(x) - P(x) = (R(x) - Q(x)) + (Q(x) - P(x))$ is divisible by x^2+1.

Therefore, $\sim$ is an equivalence relation on $\mathbb{R}[x]$.

(b) By division algorithm, $P(x) = (x^2+1)q(x) + (ax+b)$ for some $q(x) \in \mathbb{R}[x]$ and for some $a, b \in \mathbb{R}$.
Then, $P(x) - (ax-b) = (x^2+1)q(x)$, so $(ax+b) \sim P(x)$ and $[ax+b] = [P(x)]$.

(c) $[ax+b] = [cx+d] \Leftrightarrow (ax+b) \sim (cx+d) \Leftrightarrow (cx+d) - (ax+b)$ is divisible by x^2+1.
However, $(cx+d) - (ax+b) = (c-a)x + (d-b)$ is just a linear polynomial and it is divisible by x^2+1 if and only if it is a zero polynomial, i.e. $a = c$ and $b = d$.

(d) Note that every equivalence class in $\mathbb{R}[x]/\sim$ is in form of $[P(x)]$ where $P(x) \in \mathbb{R}[x]$. However, from (b) and (c), we know that for each equivalence class $[P(x)]$ there is one and only one linear polynomial $ax+b$ such that $[P(x)] = [ax+b]$. Therefore, $\mathbb{R}[x]/\sim = \{[ax+b] : a, b \in \mathbb{R}\}$.

(e) The proof is similar to 2(a).

(f) $[2x+3] \cdot [3x+1] = [6x^2+11x+3] = [6(x^2+1) + (11x-3)] = [11x-3]$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^3$. Show that $f(x)$ is injective.

Ans: Suppose that $f(x_1) = f(x_2)$. Then,

$$\begin{aligned} x_1^3 &= x_2^3 \\ x_1^3 - x_2^3 &= 0 \\ (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) &= 0 \end{aligned}$$

Note that $x_1^2+x_1x_2+x_2^2 = (x_1+\frac{1}{2}x_2)^2 + \frac{3}{4}x_2^2 \geq 0$. If $x_1^2+x_1x_2+x_2^2 = 0$, then we have $x_1 = x_2 = 0$. If $x_1^2+x_1x_2+x_2^2 \neq 0$, then we have $x_1 - x_2 = 0$, i.e. $x_1 = x_2$.

Therefore, $x_1 = x_2$ and f is injective.

(Remark: We cannot say $x_1^3 = x_2^3$ and $x_1 = \sqrt[3]{x_1^3} = \sqrt[3]{x_2^3} = x_2$ since $\sqrt[3]{x}$ is the inverse function of x^3 which is known to exist after showing x^3 is bijective.)

5. Let $f : (0, \infty) \to \mathbb{R}$ be a function such that $f'(x) > 0$ for all $x \in (0, \infty)$.

   (a) Show that f is an injective function.

   (b) Show that f may not be a surjective function by giving a counterexample.

   Ans:

   (a) By the assumption, f is strictly increasing on $(0, \infty)$, i.e. if $0 < x_1 < x_2$, then $f(x_1) < f(x_2)$. Therefore, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

   (b) Let $f : (0, \infty) \to \mathbb{R}$ which is defined by $f(x) = 1 - \dfrac{1}{x}$. Then, $f'(x) = \dfrac{1}{x^2} > 0$, but $f(x) < 1$ for all $x \in (0, \infty)$, so it is not a surjective function.

6. Let A, B and C be subset of $\mathbb{R}$, and let $g : A \to B$ and $f : C \to \mathbb{R}$ be two bijective functions such that $B \subseteq C$.

   (a) Show that the composite function $(f \circ g) : A \to \mathbb{R}$ (i.e. $(f \circ g)(x) = f(g(x))$) is injective.

   (b) Is it true that $f \circ g$ is bijective?

   Ans:

   (a) Suppose that $(f \circ g)(x_1) = (f \circ g)(x_2)$. Then,

$$
\begin{aligned}
f(g(x_1)) &= f(g(x_2)) \\
g(x_1) &= g(x_2) \qquad (\because f \text{ is injective}) \\
x_1 &= x_2 \qquad (\because g \text{ is injective})
\end{aligned}
$$

   Therefore, $f \circ g$ is injective.

   (b) Suppose that $g : [0, \infty) \to [0, \infty)$ is defined by $g(x) = \sqrt{x}$ and $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$.

   Then $f \circ g : [0, \infty) \to \mathbb{R}$ is defined by $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \sqrt{x}$ which is not surjective.