THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5510 Foundation of Advanced Mathematics 2017-2018
Suggested Solution to Quiz 2

1. (a) Find $\gcd(3626, 1274)$.
   (b) Find two integers $m$, $n$ such that $3626m + 1274n = \gcd(3626, 1274)$.

**Ans:**

(a) We have

$$\begin{aligned} 3626 &= 1274 \times 2 + 1078 \\ 1274 &= 1078 \times 1 + 196 \\ 1078 &= 196 \times 5 + 98 \\ 196 &= 98 \times 2 \end{aligned}$$

Therefore, $\gcd(3626, 1274) = 98$.

(b) By extended Euclidean algorithm, we have

$$\begin{aligned} 98 &= 1078 - 196 \times 5 \\ &= 1078 - (1274 - 1078) \times 5 \\ &= 1078 \times 6 - 1274 \times 5 \\ &= (3626 - 1274 \times 2) \times 6 - 1274 \times 5 \\ &= 3626 \times 6 - 1274 \times 17 \\ &= 3626 \times 6 + 1274 \times (-17) \end{aligned}$$

2. (a) Let $A$ and $B$ be two sets.
   State the definition of $|A| = |B|$, i.e. $A$ and $B$ are having the same cardinality.
   (b) Let $I = (0, 1)$ and let $\mathbb{R}^+$ be the set of all positive real numbers.
   By considering the function $f : (0, 1) \to \mathbb{R}^+$ defined by $f(x) = \dfrac{1}{x} - 1$, show that $|I| = |\mathbb{R}^+|$.
   (c) Give an example of sequence of sets $P_i$, $i = 1, 2, 3, \ldots$, such that $P_i$ is a proper subset of $P_{i+1}$ for all positive integers $i$, but all $P_i$ are having the same cardinality.

**Ans:**

(a) $|A| = |B|$ if **there exists** a bijective function $f : A \to B$.

(b) Let $f : (0, 1) \to \mathbb{R}^+$ be a function defined by $f(x) = \dfrac{1}{x} - 1$ and we claim that $f$ is a bijective function.

- Suppose that $x_1, x_2 \in (0, 1)$ and $f(x_1) = f(x_2)$. Then,

$$\begin{aligned} \frac{1}{x_1} - 1 &= \frac{1}{x_2} - 1 \\ \frac{1}{x_1} &= \frac{1}{x_2} \\ x_1 &= x_2 \end{aligned}$$

  Therefore, $f$ is injective.

- Let $y \in \mathbb{R}^+$. Let $x = \dfrac{1}{1+y}$.

  Note that $y > 0$, then $1 + y > 1$ and so $0 < x = \dfrac{1}{1+y} < 1$, i.e. $x \in I$.

  Also, $f(x) = f(\dfrac{1}{1+y}) = \dfrac{1}{\left(\frac{1}{1+y}\right)} - 1 = (1+y) - 1 = y$.

  Therefore, $f$ is surjective.

Therefore, $f$ is an bijective function and $|I| = |\mathbb{R}^+|$.

(c) For any positive integer $i$, define $P_i = (0, i)$, i.e. the set $\{x \in \mathbb{R} : 0 < x < i\}$.

Remark: Clearly, $P_i$ is a proper subset of $P_{i+1}$ for all positive integers $i$.

Furthermore, let $f : (0, i) \to (0, i+1)$ defined by $f(x) = \frac{i+1}{i}x$.

You may show that $f$ is a bijective function.

3. (a) Recall that the natural number 0 is defined as the empty set $\phi$, 1 is defined as $0^+$, 2 is defined as $1^+$ and etc.

   Write down the natural numbers 1, 2 and 3 as sets. Hence, explain why $1 \le 3$.

   (b) By using the definition of addition and multiplication of natural numbers, show that $1 + 1 = 2$ and $1 \times 1 = 1$.

**Ans:**

(a) Recall that $x^+$ is defined as $x \cup \{x\}$ and 0 is defined as the empty set $\phi$. Then,

$$\begin{aligned} 1 &= 0^+ = \phi \cup \{\phi\} = \{\phi\} \\ 2 &= 1^+ = \{\phi\} \cup \{\{\phi\}\} = \{\phi, \{\phi\}\} \\ 3 &= 2^+ = \{\phi, \{\phi\}\} \cup \{\{\phi, \{\phi\}\}\} = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\} \end{aligned}$$

We can see that 1 is a set that contains one element $\phi$, which is also an element in 3. Therefore, 1 is a subset of 3. By definition, $1 \le 3$.

Remark: By definition, for any natural numbers $m$ and $n$, $m \le n$ if $m$ is a subset of $n$.

(b) $1 + 1 = 1 + 0^+ = (1 + 0)^+ = 1^+ = 2$ and

$1 \times 1 = 1 \times 0^+ = 1 \times 0 + 1 = 0 + 1 = 0 + 0^+ = (0 + 0)^+ = 0^+ = 1$.

4. Let $m, n$ be two natural numbers. Recall the fact that $m^+ = n^+$ implies $m = n$.

   Suppose that $x, y$ be two natural numbers such that $y + x = x$. Prove that $y = 0$.

   (Hint: Prove by mathematical induction on $x$.)

**Ans:**

- When $x = 0$, if $y + x = x$, which means $y + 0 = 0$ and so $y = 0$.
- Assume that $x$ is a natural number such that if $y + x = y$ then $y = 0$.

  If $y + x^+ = x^+$, then

$$\begin{aligned} y + x^+ &= x^+ \\ (y + x)^+ &= x^+ \\ y + x &= x \\ y &= 0 \qquad \text{(By assumption)} \end{aligned}$$

By mathematical induction, for any two natural numbers $x$ and $y$, if $y + x = x$, then $y = 0$.