THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5510 Foundation of Advanced Mathematics 2017-2018
Suggested Solution to Quiz 1

1. Let P, Q and R be three statements. By constructing truth tables, show that

   (a) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$;

   (b) $P \to Q \equiv (\neg Q) \to (\neg P)$.

   Ans:

   (a)

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

   From the fifth column and the last column, we have $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$.

   (b)

P	Q	$P \to Q$	$\neg Q$	$\neg P$	$(\neg Q) \to (\neg P)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

   From the third column and the last column, we have $P \to Q \equiv (\neg Q) \to (\neg P)$.

2. Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a function.

   (a) f is said to be a positive function if

   - for all $x \in A$, $f(x) > 0$.

   (b) f is said to be bounded above if

   - there exists $M \in \mathbb{R}$ such that for all $x \in A$, $f(x) \leq M$.

   (c) $\lim_{x \to +\infty} f(x) = +\infty$ if

   - for all $M > 0$, there exists $N \in \mathbb{R}$ such that for all $x \geq N$, $f(x) \geq M$.

   Write down the **negation** of the above definitions.

   Ans:

   (a) f is not a positive function if there exists $x \in A$ such that $f(x) \leq 0$.
   $(\exists x \in A)(f(x) \leq 0)$

   (b) f is not bounded above if for all $M \in \mathbb{R}$, there exists $x \in A$ such that $f(x) > M$.
   $(\forall M \in \mathbb{R})(\exists x \in A)(f(x) > M)$

   (c) $\lim_{x \to +\infty} f(x) \neq +\infty$ if there exists $M > 0$ such that for all $N \in \mathbb{R}$ there exists $x \geq N$ such that $f(x) < M$.
   $(\exists M > 0)(\forall N \in \mathbb{R})(\exists x \geq N)(f(x) < M)$

3. Prove that the sum of a rational number and an irrational number is an irrational number.

Ans:

Suppose the contrary, i.e. there exists a rational number x and an irrational number y such that the sum z is a rational number.

We let $x = \dfrac{m}{n}$ and $z = \frac{p}{q}$ where m, n, p, q are integers and $n, q \neq 0$. Then,

$$\begin{aligned} x + y &= z \\ \frac{m}{n} + y &= \frac{p}{q} \\ y &= \frac{p}{q} - \frac{m}{n} \\ &= \frac{np - mq}{qn} \end{aligned}$$

Since m, n, p, q are integers, $np - mq$ and qn are integers. It means y is a rational number, which is a contradiction.

Therefore, the sum of a rational number and an irrational number is an irrational number.

4. Define a relation $\sim$ on $\mathbb{Z}$ such that $a \sim b$ if and only if $b - a$ is divisible by 7.

   (a) Show that $\sim$ defines an equivalence relation.

   (b) Show that the addition $+$ on $\mathbb{Z}$ induces an addition $\boxplus$ on $\mathbb{Z}_7 = \mathbb{Z}/\sim$.

   (c) Show that the addition $\boxplus$ on $\mathbb{Z}_7$ is associative.

   (d) Evaluate $[9] \boxplus [11]$, where $[9], [11] \in \mathbb{Z}_7$.

Ans:

(a) i. (Reflexive) Let $a \in \mathbb{Z}$. Then, $a - a = 0$ which is divisible by 7 and so $a \sim a$.

ii. (Symmetric) Let $a, b \in \mathbb{Z}$ such that $a \sim b$.
Then $b - a$ is divisible by 7, i.e. $b - a = 7M$ for some integer M.
We have $a - b = -7M = 7(-M)$, where $-M$ is an integer. Therefore, $a - b$ is divisible by 7 and so $b \sim a$.

iii. (Transitive) Let $a, b, c \in \mathbb{Z}$ such that $a \sim b$ and $b \sim c$.
Then $b - a$ and $c - b$ are divisible by 7, i.e. $b - a = 7M$ and $c - b = 7N$ for some integers M, N.
We have $c - a = (c - b) + (b - a) = 7(M + N)$, where $M + N$ is an integer. Therefore, $c - a$ is divisible by 7 and so $c \sim a$.

Therefore, $\sim$ is an equivalence relation on $\mathbb{Z}$.

(b) Let $a, b, a', b' \in \mathbb{Z}$ such that $a \sim a'$ and $b \sim b'$.
Then $a' - a$ and $b' - b$ are divisible by 7, $a' - a = 7M$ and $b' - b = 7N$ for some integers M, N.
We have $(a' + b') - (a + b) = (a' - a) + (b' - b) = 7(M + N)$, where $M + N$ is an integer. Therefore, $(a' + b') - (a + b)$ is divisible by 7 and so $(a + b) \sim (a' + b')$.
Therefore, the addition $+$ on $\mathbb{Z}$ induces an addition $\boxplus$ on $\mathbb{Z}_7 = \mathbb{Z}/\sim$.

(c) Let $[a], [b], [c] \in \mathbb{Z}_7$, where $a, b, c \in \mathbb{Z}$. Then,

$$\begin{aligned} ([a] \boxplus [b]) \boxplus [c] &= [a + b] \boxplus [c] \\ &= [(a + b) + c] \\ &= [a + (b + c)] \qquad \text{(associative law of + on } \mathbb{Z}) \\ &= [a] \boxplus [b + c] \\ &= [a] \boxplus ([b] \boxplus [c]) \end{aligned}$$

Therefore, the addition $\boxplus$ on $\mathbb{Z}_7$ is associative.

(d) $[9] \boxplus [11] = [9 + 11] = [20] = [6]$ (Alternative method: $[9] \boxplus [11] = [2] \boxplus [4] = [2 + 4] = [6]$)