THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5510 Foundation of Advanced Mathematics 2017-2018
Suggested Solution to Assignment 2

1. Let $x_1, x_2 \in [0, \infty)$ such that $f(x_1) = f(x_2)$. Then,

$$\begin{aligned} f(x_1) &= f(x_2) \\ x_1^2 &= x_2^2 \\ (x_1 - x_2)(x_1 + x_2) &= 0 \end{aligned}$$

We have $x_1 - x_2 = 0$ or $x_1 + x_2 = 0$. For the first case, clearly we have $x_1 = x_2$; for the second case, since $x_1, x_2 \geq 0$, it can only be $x_1 = x_2 = 0$. In both cases, we have $x_1 = x_2$, and so f is injective.

If we take $y = -1 \in \mathbb{R}$, there exists no $x \in [0, \infty)$ such that $f(x) = x^2 = -1 = y$. Therefore, f is not surjective.

2. Let $x_1, x_2 \in A$ such that $(g \circ f)(x_1) = (g \circ f)(x_2)$, i.e. $g(f(x_1)) = g(f(x_2))$.

Since g is injective, $f(x_1) = f(x_2)$. Then, since f is injective, $x_1 = x_2$.

Therefore $g \circ f$ is injective.

Let $y \in C$. Since g is surjective, there exists $w \in B$ such that $g(w) = y$.

Also, since f is surjective, there exists $x \in A$ such that $f(x) = w$.

Then, we have $(g \circ f)(x) = g(f(x)) = g(w) = y$ and so $g \circ f$ is surjective.

3. Let $y \in \mathbb{R}$. Take $a = -(1 + |y|)$ and $b = 1 + |y|$.

Note that $b^3 = 1 + 3|y| + 3|y|^2 + |y|^3 > 1 + |y| > y$ and $a^3 = -1 - 3|y| - 3|y|^2 - |y|^3 < -1 - |y| < y$. Therefore, we have $a < b$ and $f(a) < y < f(b)$.

By using the intermediate value theorem, there exists $c \in (a, b)$ such that $f(c) = y$.

4. $2 + 3 = 2 + 2^+ = (2 + 2)^+ = (2 + 1^+)^+ = ((2 + 1)^+)^+ = ((2 + 0^+)^+)^+ = (((2 + 0)^+)^+)^+ = ((2^+)^+)^+ = (3^+)^+ = 4^+ = 5$

5. (a) When $m = 0$, $0 \times m = 0 \times 0 = 0$.

Assume that $0 \times m = 0$ for $m \in \mathbb{N}$. Then,

$$0 \times m^+ = 0 \times m + 0 = 0 + 0 = 0.$$

By mathematical induction, we have $0 \times m = 0$ for all $m \in \mathbb{N}$.

(b) When $m = 0$, $1 \times m = 1 \times 0 = 0$.

Also, $m \times 1 = 0 \times 1 = 0 \times 0^+ = 0 \times 0 + 0 = 0 + 0 = 0$.

Therefore, $1 \times 0 = 0 \times 1 = 0$.

Assume that $1 \times m = m \times 1 = m$ for $m \in \mathbb{N}$. Then,

$$1 \times m^+ = 1 \times m + 1 = m + 1 = m^+.$$

(Remark: It should be already known that $m+1 = m+0^+ = (m+0)^+ = m^+$.)

$$m^+ \times 1 = m^+ \times 0^+ = m^+ \times 0 + m^+ = 0 + m^+$$

Therefore, $1 \times m^+ = m^+ \times 1 = m^+$. By mathematical induction, we have $1 \times m = m \times 1 = m$ for all $m \in \mathbb{N}$.

(c) When $n = 0$, $m^+ \times n = m^+ \times 0 = 0$ and $m \times n + n = m \times 0 + 0 = 0$.
Assume that for a particular $n \in \mathbb{N}$, we have $m^+ \times n = m \times n + n$ for all $m \in \mathbb{N}$.
Then, for all $m \in \mathbb{N}$,

$$m^+ \times n^+ = m^+ \times n + m^+ = (m \times n + n) + m^+ = m \times n + (n + m^+) = m \times n + (m + n^+) = (m \times n + m) + n^+$$

(Remark: $n + m^+ = (n+m)^+ = (m+n)^+ = m + n^+$)
By mathematical induction, we have $m^+ \times n = m \times n + n$ for all $m, n \in \mathbb{N}$.

(d) When $m = 0$, it is already known that $m \times 0 = 0 \times m = 0$ for all $m \in \mathbb{N}$.
Assume that for a particular $n \in \mathbb{N}$, we have $m \times n = n \times m$ for $m \in \mathbb{N}$.
Then, for all $m \in \mathbb{N}$,

$$m^+ \times n = m \times n + n = n \times m + n = n \times m^+$$

By mathematical induction, we have $m \times n = n \times m$ for all $m, n \in \mathbb{N}$.

(e) When $p = 0$, $m \times (n+p) = m \times (n+0) = m \times n = m \times n$ and $m \times n + m \times p = m \times n + m \times 0 = m \times n$.
Assume that for a particular $p \in \mathbb{N}$, we have $m \times (n+p) = m \times n + m \times p$ for all $m, n \in \mathbb{N}$.
Then, for all $m, n \in \mathbb{N}$,

$$m \times (n + p^+) = m \times (n^+ + p) = m \times n^+ + m \times p = (m \times n + m) + m \times p = m \times n + (m + m \times p)$$
$$= m \times n + (m \times p + m) = m \times n + m \times p^+$$

(Remark: $n + p^+ = (n+p)^+ = (p+n)^+ = p + n^+ = n^+ + p$.)
By mathematical induction, we have $m \times (n+p) = m \times n + m \times p$ for all $m, n, p \in \mathbb{N}$.

(f) When $p = 0$, $(m \times n) \times p = (m \times n) = 0$ and $m \times (n \times p) = m \times (n \times 0) = m \times 0 = 0$.
Assume that for a particular $p \in \mathbb{N}$, we have $(m \times n) \times p = m \times (n \times p)$ for all $m, n \in \mathbb{N}$.
Then, for all $m, n \in \mathbb{N}$,

$$(m \times n) \times p^+ = (m \times n) \times (p+1) = (m \times n) \times p + (m \times n) \times 1 = m \times (n \times p) + m \times n$$
$$= m \times (n \times p + n) = m \times (n \times p + n \times 1) = m \times (n \times (p+1)) = m \times (n \times p^+)$$

(Remark: $p^+ = (p+0)^+ = p + 0^+ = p + 1$.)
By mathematical induction, we have $(m \times n) \times p = m \times (n \times p)$ for all $m, n, p \in \mathbb{N}$.

6. Suppose that there exists natural numbers n and m such that $n < m < n^+$, i.e. $n < m$ and $m < n^+$.

Note that $m < n^+$, so $m \in n^+ = n \cup \{n\}$. Therefore, there are only two possible cases:

Case 1: $m \in n$, then it implies $m < n$ which contradicts to the fact that $n < m$.

Case 2: $m \in \{n\}$, then $m = n$ which again contradicts to the fact that $n < m$.

Therefore, both cases lead contradiction.

7. (a) Recall the fact that for any natural numbers m and n,

    - $m < n^+$ if and only if $m \leq n$,
    - $m^+ \leq n$ if and only if $m < n$.

    (Please refer to theorem 6.6 of The Elementary Set Theory for the statement as well as the proof.)

    Then, we have $m < n$ if and only if $m^+ \leq n$ (second statement),

    if and only if $m^+ < n^+$ (first statement but replacing m by m^+).

   (b) When $p = 0$, it's trivial. Assume that for a particular $p \in \mathbb{N}$, we have $m < n$ if and only if $m + p < n + p$ for all $m, n \in \mathbb{N}$.

    Then, for all $m, n \in \mathbb{N}$,

$$\begin{aligned} m < n &\Leftrightarrow m + p < n + p \qquad \text{(Induction assumption)} \\ &\Leftrightarrow m + p^+ = (m + p)+ < (n + p)^+ = n + p^+ \qquad \text{(By (a))} \end{aligned}$$

    By mathematical induction, we have $m < n$ if and only if $m + p < n + p$ for all $m, n, p \in \mathbb{N}$.

8. When $p = 1$, it's trivial.

   Assume that for a particular $p \in \mathbb{N}$, we have $m < n$ if and only if $mp < np$ for all $m, n \in \mathbb{N}$.

   Then, for all $m, n \in \mathbb{N}$:

   - if $m < n$, then by induction assumption, we have $mp < np$ and so $mp^+ = mp + m < np + m$. On the other hand, we have $m < n$, so $np + m = m + np < n + np = np + n = np^+$.

     Therefore, we have $mp^+ < np^+$.

   - if $mp^+ < np^+$, we are going to prove that $m < n$ by contradiction.

     Suppose the contrary and we have $n \leq m$. Then, $mp + m = mp^+ < np^+ = np + n \leq np + m$. Therefore by the previous question, we have $mp < np$ which implies $m < n$ whcih is a contradiction.

     (Remark: From the previous question, the contrapositive of the statement in (a) gives $m \leq n$ if and only if $m^+ \leq n^+$. By using mathematical induction like (b), we have $m \leq n$ if and only if $m + p \leq n + p$ for all $m, n, p \in \mathbb{N}$.)

   Therefore, $m < n$ if and only if $m + p^+ \leq n + p^+$ for all $m, n \in \mathbb{N}$. By mathematical induction, we have $m < n$ if and only if $mp < np$ for all $m, n, p \in \mathbb{N}$.